APPSEM'98

A categorical semantics of $MetaML$

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joint work with
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- multi-level languages and modal/temporal logic
  $\lambda^{\Box}$ [DP96@POPL], $\lambda^{\bigcirc}$ [Dav96@LICS]
- categorical semantics of 2-level languages
  λ_{cr}, λ_{sd} [Mog97@MFPS]
- $MetaML$: a strongly typed multi-stage language
  $MetaML$ [TS97@PEPM]
  λR [TBS98@ICALP] $= \lambda^{\bigcirc} +$ persistence $+$ **run**

* monadic metalanguage ML^{ω}_{M} with *modalities*

* categorical semantics of ML^{ω}_{M} in ω-level categories

* translation of $MetaML$ into $ML^{\omega}_{M\mathsf{PN}}$
  → better understanding of $MetaML$ type system and its relations with $\lambda^{\bigcirc}$
  → suggests extensions with *closed code types*

* computational adequacy result

Informal definitions

- multi-stage programming = generic programs + controlled specialization
- values of **code type** $\langle\tau\rangle$ = *representations* of programs of type τ

A motivating example

```
type Z = integers;
type X = integer variables;
type E = integer expr. in X;
val e: E = an expression;
val rho_i: X -> Z = valuations;

(* expr. evaluation *)
val eval: E -> (X->Z) -> Z = ...;
val f = eval e;
val it = (f rho_1, f rho_2);

(* staged expr. evaluation *)
val eval': E -> <(X->Z)->Z> = ...;
val f' = run (eval' e); <<= compile
val it = (f' rho_1, f' rho_2);
```

Idealized $MetaML$: λR [TBS98]

- $\tau \in T ::= b \mid \tau_1 \to \tau_2 \mid \langle \tau \rangle$
- $e \in Exp ::= x \mid c \mid \lambda x.e \mid e_1\ e_2 \mid \langle e \rangle \mid \tilde{}e \mid \mathsf{run}\ e$

Typing rules

- $\Gamma = \{x_i : \tau_i^{n_i} | i \in m\}$ with x_i distinct
- $\Gamma \vdash c : \tau_c^n$
- $\Gamma \vdash x : \tau^{n'}$ if $\Gamma(x) = \tau^n$ and $n \leq n'$
- $\dfrac{\Gamma, x : \tau_1^n \vdash e : \tau_2^n}{\Gamma \vdash \lambda x.e : \tau_1 \to \tau_2^n}$ $\qquad$ $\dfrac{\begin{array}{l}\Gamma \vdash e : \tau_1 \to \tau_2^n \\ \Gamma \vdash e_1 : \tau_1^n\end{array}}{\Gamma \vdash e\ e_1 : \tau_2^n}$
- $\dfrac{\Gamma \vdash e : \tau^{n+1}}{\Gamma \vdash \langle e \rangle : \langle \tau \rangle^n}$ $\qquad$ $\dfrac{\Gamma \vdash e : \langle \tau \rangle^n}{\Gamma \vdash \tilde{}e : \tau^{n+1}}$ $\qquad$ $\dfrac{\Gamma^+ \vdash e : \langle \tau \rangle^n}{\Gamma \vdash \mathsf{run}\ e : \tau^n}$

Operational semantics (with errors)

- $e \overset{n}{\Rightarrow} e'$ big-step operational semantics
- well-typed expressions cannot go wrong
  $\emptyset \vdash e : \tau^n$ and $e \overset{n}{\Rightarrow} e'$ imply $e' \not\equiv err$

* bug in reduction semantics of [TBS98]

Big-step operational semantics

- $\dfrac{\bot \overset{0}{\Rightarrow} e}{\bot \overset{0}{\Rightarrow} e}$ $\qquad \bot \overset{n+}{\Rightarrow} \bot \qquad x \overset{n+}{\Rightarrow} x$

- $\lambda x.e \overset{0}{\Rightarrow} \lambda x.e \qquad \dfrac{e \overset{n+}{\Rightarrow} e'}{\lambda x.e \overset{n+}{\Rightarrow} \lambda x.e'}$

- $\dfrac{\begin{array}{l} e_1 \overset{0}{\Rightarrow} \lambda x.e \\ e_2 \overset{0}{\Rightarrow} e_2' \quad e[x := e_2'] \overset{0}{\Rightarrow} e' \end{array}}{e_1\ e_2 \overset{0}{\Rightarrow} e'} \qquad \dfrac{\begin{array}{l} e_1 \overset{n+}{\Rightarrow} e_1' \\ e_2 \overset{n+}{\Rightarrow} e_2' \end{array}}{e_1\ e_2 \overset{n+}{\Rightarrow} e_1'\ e_2'}$

- $\dfrac{e \overset{0}{\Rightarrow} \langle e' \rangle \quad e' \overset{0}{\Rightarrow} e''}{\mathsf{run}\ e \overset{0}{\Rightarrow} e''} \qquad \dfrac{e \overset{n+}{\Rightarrow} e'}{\mathsf{run}\ e \overset{n+}{\Rightarrow} \mathsf{run}\ e'}$

- $\dfrac{e \overset{n+}{\Rightarrow} e'}{\langle e \rangle \overset{n}{\Rightarrow} \langle e' \rangle} \qquad \dfrac{e \overset{0}{\Rightarrow} \langle e' \rangle}{\tilde{}e \overset{1}{\Rightarrow} e'} \qquad \dfrac{e \overset{n+}{\Rightarrow} e'}{\tilde{}e \overset{n++}{\Rightarrow} \tilde{}e'}$

Examples of evaluation

- $\langle \tilde{}\bot \rangle : \langle \tau \rangle \overset{0}{\Rightarrow}$ and $\langle \lambda x.\tilde{}\bot \rangle : \langle \tau_1 \to \tau_2 \rangle \overset{0}{\Rightarrow}$ diverge
- $\langle \bot \rangle : \langle \tau \rangle \overset{0}{\Rightarrow} \langle \bot \rangle$ and $\langle \langle \tilde{}\bot \rangle \rangle : \langle \langle \tau \rangle \rangle \overset{0}{\Rightarrow} \langle \langle \tilde{}\bot \rangle \rangle$
- $\tilde{}\bot \overset{0}{\Rightarrow}$ and $\langle \lambda x.\tilde{}x \rangle \overset{0}{\Rightarrow}$ get stuck

*
```
f x y = if x=0 then 0 else x*y;
g x = <fn y=>~(if x=0 then <0> else <x*y>)>;
```

$f\ 0 \overset{0}{\Rightarrow} \lambda y.\text{if } x = 0 \text{ then } 0 \text{ else } x * y$

$g\ 0 \overset{0}{\Rightarrow} \langle \lambda y.0 \rangle$

Categorical semantics of $MetaML$

- $\mathcal{S}$ semantic category
  parameterized w.r.t. a base category $\mathcal{B}$

* ML^{ω}_{M} monadic metalanguage with modalities
  conservative extension of typed λ-calculus

- interpretation of ML^{ω}_{M} in $\mathcal{S}$
  parameterized w.r.t. a strong monad M over $\mathcal{B}$
  modalities *explained* by adjunctions

* translation of $MetaML$ into $ML^{\omega}_{M\mathsf{PN}}$

* computational adequacy
  $e \Downarrow_0$ iff $\bot \neq [\![e]\!] \in M[\![\tau]\!]$, when $\emptyset \vdash e{:}\, \tau^0$

Notation

- $GF = G \circ F$ and $HGF = H \circ G \circ F$
  when F, G and H functors
- $GF\_ = G(F\_)$, when $\_$ object or morphism

The semantic category $\mathcal{S}$

- $\mathcal{B}$ CCC with small limits, e.g. **Set** or **Cpo**

* $\mathcal{S} \triangleq \mathcal{B}^{(\omega^{op})}$ *ω-levels* category (CCC)

$$
\begin{array}{lllllll}
\omega^{op} & 0 \longleftarrow & 1 \longleftarrow & 2 & \dots & n \longleftarrow & n+1 \quad \dots
\end{array}
$$

$$
\begin{array}{ccccccccc}
A & A_0 & \xleftarrow{a_0} & A_1 & \xleftarrow{a_1} & A_2 & \dots & A_n & \xleftarrow{a_n} & A_{n+1} & \dots \\
\downarrow f & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & \dots & \downarrow f_n & & \downarrow f_{n+1} & \dots \\
B & B_0 & \xleftarrow[b_0]{} & B_1 & \xleftarrow[b_1]{} & B_2 & \dots & B_n & \xleftarrow[b_n]{} & B_{n+1} & \dots
\end{array}
$$

- M strong monad over $\mathcal{B}$ (basic computations), e.g. $MX = X + 1$ or $MX = X_\perp$

A typed metalanguage: $ML^{\omega}_{M\mathsf{PNB}}$

$$
\begin{array}{rll}
\tau \in T ::= & b \mid (\tau_i \mid i \in m) \mid \tau_1 \to \tau_2 \mid & \\
& \mid M\tau & \text{comp. types} \\
& \mid \mathsf{P}\tau \mid \mathsf{N}\tau & \text{open code} \\
& \mid \mathsf{B}\tau & \text{closed code}
\end{array}
$$

$$
\begin{array}{llllllll}
MA = & MA_0 \xleftarrow{Ma_0} & MA_1 \xleftarrow{Ma_1} & MA_2 & \dots & MA_n \xleftarrow{Ma_n} & MA_{n+1} & \dots \\
\mathsf{N}A = & 1 \xleftarrow{!} & A_0 \xleftarrow{a_0} & A_1 & \dots & A_{n-1} \xleftarrow{a_{n-1}} & A_n & \dots \\
\mathsf{P}A = & A_1 \xleftarrow{a_1} & A_2 \xleftarrow{a_2} & A_3 & \dots & A_{n+1} \xleftarrow{a_{n+1}} & A_{n+2} & \dots \\
\mathsf{B}A = & \lim A \xleftarrow{id} & \lim A \xleftarrow{id} & \lim A & \dots & \lim A \xleftarrow{id} & \lim A & \dots
\end{array}
$$

Categorical properties of $\mathcal{S}$

- reflections, coreflections, and lim. pres. functors

$$\begin{tikzcd}[column sep=huge, row sep=large]
\mathcal{B} \arrow[r, hook, "\Delta" description] & \mathcal{S} \arrow[l, dashed, bend right=50, "\pi"'] \arrow[l, dashed, bend left=50, "\Gamma"]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge]
\mathcal{S} \arrow[r, hook, bend right=20, "\mathsf{N}"'] \arrow[r, phantom, "\perp"] & \mathcal{S} \arrow[l, dashed, bend right=20, "\mathsf{P}"']
\end{tikzcd}$$

- $\mathsf{B} = \Delta \circ \Gamma$ idempotent (limit preserving) comonad
- $\mathsf{N}(Y^X) \to (\mathsf{N}Y)^{\mathsf{N}X}$ iso, since P preserve limits
- *orthogonality*: $\Delta Y \to (\Delta Y)^{\mathsf{N}X}$ iso
- $up_X \colon X \to \mathsf{N}X$ s.t. $up_{\mathsf{N}X} = \mathsf{N}\; up_X$

$$up_X = \quad \begin{tikzcd}
X_0 \arrow[d, "!"] & X_1 \arrow[l, "x_0"'] \arrow[d, "x_0"] & X_2 \arrow[l, "x_1"'] \arrow[d, "x_1"] & \dots & X_n \arrow[d, "x_{n-1}"] & X_{n+1} \arrow[l, "x_n"'] \arrow[d, "x_n"] & \dots \\
1 & X_0 \arrow[l, "!"'] & X_1 \arrow[l, "x_0"'] & \dots & X_{n-1} & X_n \arrow[l, "x_{n-1}"'] & \dots
\end{tikzcd}$$

- $\delta p_X \colon M\mathsf{P}X \to \mathsf{P}MX$ iso, $\delta n_X \colon M\mathsf{N}X \to \mathsf{N}MX$

$$\delta n_X = \quad \begin{tikzcd}
M1 \arrow[d, "!"] & MX_0 \arrow[l, "M!"'] \arrow[d, "id"] & MX_1 \arrow[l, "Mx_0"'] \arrow[d, "id"] & \dots & MX_{n-1} \arrow[d, "id"] & MX_n \arrow[l, "Mx_{n-1}"'] \arrow[d, "id"] & \dots \\
1 & MX_0 \arrow[l, "!"'] & MX_1 \arrow[l, "Mx_0"'] & \dots & MX_{n-1} & MX_n \arrow[l, "Mx_{n-1}"'] & \dots
\end{tikzcd}$$

$(\mathsf{P}, \delta p)$ and $(\mathsf{N}, \delta n)$ monad endomorphisms on $(\mathcal{S}, M)$

From λR to $ML^{\omega}_{M\mathsf{PN}}$

- translation of types

$\tau \in \lambda R$	$\tau* \in ML^{\omega}_{M\mathsf{PN}}$
b	b
$\tau_1 \to \tau_2$	$\tau_1* \to M\tau_2*$
$\langle\tau\rangle$	$\mathsf{N}M\tau*$

* translation of terms

$$\Gamma \equiv \{x_i : \tau_i^{n_i} \mid i \in m\} \vdash e : \tau^n$$

$$\Gamma* \equiv \{x_i : \ \mathsf{N}^{n_i}\ \tau_i* \mid i \in m\} \vdash ? : \tau\star^n$$

where $\tau\star^n \triangleq (M\mathsf{N})^n\ M\ \tau*$

$$\tau* = \quad A_0 \xleftarrow{a_0} A_1 \xleftarrow{a_1} A_2 \quad \ldots \quad A_n \xleftarrow{a_n} A_{n+1} \quad \ldots$$

$$\tau\star^n = \quad M1 \xleftarrow{M!} M^2 1 \xleftarrow{M^2!} M^3 1 \quad \ldots \quad M^{n+1}A_0 \xleftarrow{M^{n+1}a_0} M^{n+1}A_1 \quad \ldots$$

Auxiliary morphisms

- $\perp^{\star}_n : 1 \to (M\mathsf{N})^n MA$
- $up^{\star}_{m,n} : \mathsf{N}^m A \to (M\mathsf{N})^n MA$ provided $m \le n$
- $@^{\star}_n : (M\mathsf{N})^n M\ (MB)^A \times (M\mathsf{N})^n MA \to (M\mathsf{N})^n MB$

* $\lambda^{\star}_n : ((M\mathsf{N})^n MB)^{\mathsf{N}^n A} \to (M\mathsf{N})^n M\ (MB)^A$

- $run^{\star}_n : \mathsf{P}(M\mathsf{N})^{n+1} MA = M(M\mathsf{N})^n MA \to (M\mathsf{N})^n MA$

Interpretation of λR in $\mathcal{S}$

$$[\![\Gamma \vdash \bot : \tau^n]\!] \triangleq\ !; \bot_n^\star : X \to (M\mathsf{N})^n MA$$

$$[\![\Gamma \vdash x : \tau^n]\!] \triangleq \pi_x ; up^\star_{n_x, n} : X \to (M\mathsf{N})^n MA$$

$$\frac{[\![\Gamma, x : \tau^n \vdash e : \tau'^n]\!] = f : X \times \mathsf{N}^n A \to (M\mathsf{N})^n MB}{[\![\Gamma \vdash \lambda x.e : \tau \to \tau'^n]\!] \triangleq \Lambda f ; \lambda_n^\star : X \to (M\mathsf{N})^n M\ (MB)^A}$$

$$\frac{\begin{array}{l}[\![\Gamma \vdash e_1 : \tau \to \tau'^n]\!] = f_1 : X \to (M\mathsf{N})^n M\ (MB)^A \\ [\![\Gamma \vdash e_2 : \tau^n]\!] = f_2 : X \to (M\mathsf{N})^n MA\end{array}}{[\![\Gamma \vdash e_1\ e_2 : \tau'^n]\!] \triangleq \langle f_1, f_2 \rangle ; @_n^\star : X \to (M\mathsf{N})^n MB}$$

$$\frac{[\![\Gamma \vdash e : \tau^{n+1}]\!] = f : X \to (M\mathsf{N})^{n+1} MA}{[\![\Gamma \vdash \langle e \rangle : \langle \tau \rangle^n]\!] \triangleq f : X \to (M\mathsf{N})^n M(\mathsf{N}MA)}$$

$$\frac{[\![\Gamma \vdash e : \langle \tau \rangle^n]\!] = f : X \to (M\mathsf{N})^n M(\mathsf{N}MA)}{[\![\Gamma \vdash \tilde{}e : \tau^{n+1}]\!] \triangleq f : X \to (M\mathsf{N})^{n+1} MA}$$

$$\frac{[\![\Gamma^+ \vdash e : \langle \tau \rangle^n]\!] = f : \mathsf{N}X \to (M\mathsf{N})^n M(\mathsf{N}MA)}{[\![\Gamma \vdash \mathsf{run}\ e : \tau^n]\!] \triangleq \mathsf{P}f ; run_n^\star : X \to (M\mathsf{N})^n MA}$$

$$X = \mathsf{PN}X$$

where $A \triangleq [\![\tau *]\!]$, $B \triangleq [\![\tau' *]\!]$ and $X \triangleq [\![\Gamma *]\!]$

Definition of auxiliary morphisms

$\perp^\star_n$) $1 \xrightarrow{\quad\perp\quad} MA \xrightarrow{\ell^n} (M\mathsf{N})^n MA$

where $\ell = up; \eta \colon A \to M\mathsf{N}A$

$up^\star_{0,n}$) $A \xrightarrow{\quad\eta\quad} MA \xrightarrow{\ell^n} (M\mathsf{N})^n MA$

$up^\star_{k,n+k}$) $\mathsf{N}^k A \xrightarrow{L^k(up^\star_{0,n})} (M\mathsf{N})^{n+k} MA$

where $Lf = \mathsf{N}f; \eta \colon \mathsf{N}A \to M\mathsf{N}B$ when $f \colon A \to B$

$@^\star_0$) $(MB)^A \times MA \to MB$ usual CBV translation

$@^\star_{n+1}$) $(M\mathsf{N})^{n+1}M\ (MB)^A \times (M\mathsf{N})^{n+1}MA \xrightarrow{\qquad\psi'\qquad}$

$M\mathsf{N}((M\mathsf{N})^n M\ (MB)^A \times (M\mathsf{N})^n MA) \xrightarrow{\quad M\mathsf{N}\ @^\star_n\quad}$

$(M\mathsf{N})^{n+1}MB$

$\psi' \colon M\mathsf{N}A \times M\mathsf{N}B \to M(\mathsf{N}A \times \mathsf{N}B) = M\mathsf{N}(A \times B)$

$run^\star_0$) $\mu \colon M^2 A \to MA$

$run^\star_{n+1}$) $M(M\mathsf{N})^{n+1}MA \xrightarrow{\qquad M(\delta n)\qquad}$

$M\mathsf{N}M(M\mathsf{N})^n MA \xrightarrow{M\mathsf{N}(run^\star_n)} (M\mathsf{N})^{n+1}MA$

Definition of auxiliary morphisms

$\lambda^\star_0$) $\eta\colon (MB)^A \to M\ (MB)^A$ usual CBV translation

$\lambda^\star_{n+1}$) $((M\mathsf{N})^{n+1}MB)^{\mathsf{N}^{n+1}A} \xrightarrow{\quad\lambda^\star\quad}$

$M\mathsf{N}\ ((M\mathsf{N})^n MB)^{\mathsf{N}^n A} \xrightarrow{\quad M\mathsf{N}(\lambda^\star_n)\quad}$

$(M\mathsf{N})^{n+1}M\ (MB)^A$

where $\lambda^\star\colon (M\mathsf{N}MB)^{\mathsf{N}A} \to M\mathsf{N}\ (MB)^A$

f) $(M\mathsf{N}MB)^{\mathsf{N}A} \xrightarrow{\delta n^{\mathsf{N}A}} (\mathsf{N}M^2B)^{\mathsf{N}A} \cong \mathsf{N}\ (M^2B)^A \xrightarrow{\mathsf{N}(\mu^A)} \mathsf{N}\ (MB)^A$

because $(\mathsf{N}B)^{\mathsf{N}A} \cong \mathsf{N}(B^A)$

c) $(M\mathsf{N}MB)^{\mathsf{N}A} \xrightarrow{(M!)^{\mathsf{N}A}} (M1)^{\mathsf{N}A} \cong M1$

because $(\Delta B)^{\mathsf{N}A} \cong \Delta B$

$\lambda^\star$) $(M\mathsf{N}MB)^{\mathsf{N}A} \xrightarrow{\langle c, f\rangle} M1 \times \mathsf{N}\ (MB)^A \xrightarrow{t} M\mathsf{N}\ (MB)^A$

- $\lambda^\star_0(h) = \perp$ never, where $h \in (MB)^A$
- $\lambda^\star_1(h) = \perp$ iff $c(h) = \perp$, where $h \in (M\mathsf{N}MB)^{\mathsf{N}A}$

Conclusions

- expressive metalanguage ML^{ω}_{M}
- with several modalities P, N, B, ...
- a categorical semantics in $\mathcal{B}^{\omega^{op}}$, where computational types interpreted by pointwise monad interpretation of modalities given by adjunctions
- simplification to $MetaML$ type system $\{x_i\colon \tau_i^{n_i} | i \in m\} \vdash e\colon \tau^n$

Computationally adequate translations

$* \; MetaML \to ML^{\omega}_{M\mathsf{PN}} \qquad \mathcal{S} = \mathcal{B}^{\omega^{op}}$

$? \; \lambda^{\bigcirc} \to ML^{\omega}_{M\mathsf{N}} \qquad \mathcal{S} = \mathcal{B}^{N}$

$? \; \lambda^{\Box} \to ML^{\omega}_{M\mathsf{B}} \qquad \mathcal{S} = \mathcal{B}^{2^{op}}$

Expected fallouts

- extension of $MetaML$, e.g. with closed code types
- equational calculus (fix bug in reduction of [TBS98])