Relating Models of Impredicative Type Theories

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Abstract

The object of study of this paper is the categorical semantics of three impredicative type theories, viz. Higher Order λ-calculus $F\omega$, the Calculus of Constructions and Higher Order ML. The latter is particularly interesting because it is a two-level type theory with type dependency at both levels. Having described appropriate categorical structures for these calculi, we establish translations back and forth between all of them. Most of the research in the paper concerns the theory of fibrations and comprehension categories.

1. Introduction

In recent years there has been a considerable amount of work in using category theory to analyse and model type theories. Most of this work has taken inspiration from Categorical Logic (especially Hyperdoctrines), and has developed in two directions: the modelling of Higher order Lambda Calculus $F\omega$ using indexed categories (see Seely [1987]), and the modelling of Martin-Löf Type Theory *TT* (see Cartmell [1987], Seely [1984] and Taylor [1987]) and the Calculus of Constructions *CC* (see Ehrhard [1988], Hyland & Pitts [1989] and Streicher [1989]). More recently new type theories have been proposed, *HML* (see Moggi [1991]) and the Theory of Predicates (see Pavlović [1990]), exhibiting a two-level structure (like in $F\omega$), where the first level (constructors of $F\omega$) is *independent* from the second level (terms of $F\omega$), and type dependency (like in *TT*) but now at both levels. *HML* is an example of such a type theory.

This paper introduces comprehension categories (see also Jacobs [1990]), which generalise D-categories (see Ehrhard [1988]) and classes of display maps (see Hyland & Pitts [1989]), and explains how they can be used (together with fibrations) to give a categorical treatment of type theories. Comprehension categories are unnecessarily general for describing type dependency (see Pitts [1989], Moggi [1991] for similar remarks on D-categories). In practice one uses comprehension categories which are either *cartesian* (corresponding to Cartmell's categories with attributes) or *full* (corresponding to Taylor's relative slice categories). However, most of the relevant definitions and results make sense for comprehension categories in general. Our investigation is divided in two steps:

- general results on fibrations and comprehension categories; this is "category theory over a base *category*".

- the *fibred* version of such results where the base category has been replaced by a fibration. This is "category theory over a fibration". The choice of fibrations and comprehension categories as conceptual framework for type theories makes this passage from the first to the second step particularly straightforward.

We apply some of the above results to clarify the relation between categorical models of $F\omega$, *HML* and *CC*. In comparison with other approaches to type theories ours is particularly suitable to describe type theories with two levels (or more):

- Seely [1987] considers only a two-level type theory without type dependency;

- Hyland & Pitts [1989] considers type theories with type dependency, but only one level (e.g. in *CC* constructors and terms are interdependent);

- Moggi [1991] models two-level type theories without using explicitly comprehension categories over a fibration, but then the less natural concept of *independence* (of a comprehension category from another comprehension category over the same category) has to be used;

- Pavlović [1990] models two level-type theories using classes of display maps (over a fibration). However, the use of fibrations and comprehension categories provides in our opinion great conceptual clarity.

We do realise that the use of fibred category theory makes the paper rather technical. However, whenever possible we try to give underlying type theoretical intuitions. At first reading it might be of help to concentrate on sections 2,6 and 7 and take a brief look only at the first parts of the more technical sections 3,4 and 5.

2. Summary of relations

In this section we give an informal *type-theoretic* formulation of the results in section 7 on the relations among $F\omega$ (Higher order Lambda Calculus), *HML* (Higher order ML) and *CC* (Calculus of Constructions). Type-theoretically $F\omega$, *CC* and *HML* are described by a set of rules for deriving well-formation and equality judgements. The informal description below involves only well-formation judgements and we will use the following notational conventions: k for kinds, u for constructors, v for constructor variables, τ for types, e for terms, x for term variables, Δ for constructor contexts "$v_1 : k_1, \ldots, v_m : k_m$", Γ for term contexts "$x_1 : \tau_1, \ldots, x_n : \tau_n$" and Φ for mixed contexts (allowed only in *CC*).

- $F\omega$ Higher order Lambda Calculus

well-formation judgements			
constructors		*terms*	
$\Delta \vdash_{F\omega}$	*context*	$\Delta; \Gamma \vdash_{F\omega}$	*context*
$\vdash_{F\omega} k$	*kind*	$\Delta \vdash_{F\omega} \tau$	*type*
$\Delta \vdash_{F\omega} u : k$	*constructor*	$\Delta; \Gamma \vdash_{F\omega} e : \tau$	*term*

context extension rules

$$\frac{\Delta \vdash_{F\omega} \quad \vdash_{F\omega} k}{\Delta, v : k \vdash_{F\omega}} \qquad \frac{\Delta; \Gamma \vdash_{F\omega} \quad \Delta \vdash_{F\omega} \tau}{\Delta; \Gamma, x : \tau \vdash_{F\omega}}$$

The judgements for $F\omega$ tell us that:

- kinds do not depend on variables;
- constructors and types may depend only on constructor variables;
- terms may depend both on constructor and on term variables.

Therefore the rules for kinds and constructors can be given independently from those for types and terms, and there are no dependent kinds or types.

- *HML* Higher order ML

well-formation judgements			
constructors		*terms*	
$\Delta \vdash_{HML}$	*context*	$\Delta; \Gamma \vdash_{HML}$	*context*
$\Delta \vdash_{HML} k$	*kind*	$\Delta; \Gamma \vdash_{HML} \tau$	*type*
$\Delta \vdash_{HML} u : k$	*constructor*	$\Delta; \Gamma \vdash_{HML} e : \tau$	*term*

context extension rules

$$\frac{\Delta \vdash_{HML} k}{\Delta, v : k \vdash_{HML}} \qquad \frac{\Delta; \Gamma \vdash_{HML} \tau}{\Delta; \Gamma, x : \tau \vdash_{HML}}$$

The judgements for *HML* tell us that:

- kinds and constructors may depend only on constructor variables;
- types and terms may depend both on constructor and on term variables.

As for $F\omega$ the rules for kinds and constructors can be given independently from those for types and terms, but there may be dependent kinds and types.

- *CC* Calculus of Constructions

well-formation judgements			
constructors		*terms*	
$\Phi \vdash_{CC}$	*context*	$\Phi \vdash_{CC}$	*context*
$\vdash_{CC} k$	*kind*	$\Phi \vdash_{CC} \tau$	*type*
$\Phi \vdash_{CC} u : k$	*constructor*	$\Phi \vdash_{CC} e : \tau$	*term*

context extension rules

$$\frac{\Phi \vdash_{CC} k}{\Phi, v : k \vdash_{CC}} \qquad \frac{\Phi \vdash_{CC} \tau}{\Phi, x : \tau \vdash_{CC}}$$

The judgements for *CC* are all interdependent, and there may be all sorts of dependency between kinds and types.

The relations established in section 7 are summarised by the following picture, where an arrow $S_1 \to S_2$ is a mapping from S_1-models/-theories to S_2-models/-theories. Alternatively, an arrow $S_1 \to S_2$ can be viewed as a translation from S_2 to S_1.

$$
\begin{tikzcd}
F\omega & CC \arrow[l, "U_1"'] \arrow[d, "\tilde{\ }"] & F\omega \arrow[l, "\cdot"'] \\
F\omega \arrow[u, "id"] \arrow[r, "\bar{\ }"'] & HML \arrow[ul, "U_2"] &
\end{tikzcd}
$$

We describe an arrow $S_1 \to S_2$ as a function f from S_1-theories to S_2-theories, by giving the judgements in $f(T)$ in terms of the judgements in T, where T is an S_1-theory. We consider only few *key* judgements.

- $\bar{\ }: F\omega \to HML$ is essentially an inclusion, i.e.
  - $\Delta \vdash_{HML} u : k$ iff $\Delta \vdash_{F\omega} u : k$
  - $\Delta; \Gamma \vdash_{HML} e : \tau$ iff $\Delta; \Gamma \vdash_{F\omega} e : \tau$

  so k does not depend on Δ and τ does not depend on Γ.

- $U_2 : HML \to F\omega$ is removal of dependency, i.e.
  - $u_1 : k_1, \ldots, u_m : k_m \vdash_{F\omega}$ iff $\emptyset \vdash_{HML} k_i$ for $1 \le i \le m$
  - $\Delta \vdash_{F\omega} u : k$ iff $\Delta \vdash_{F\omega}$, $\emptyset \vdash_{HML} k$ and $\Delta \vdash_{HML} u : k$
  - $\Delta; x_1 : \tau_1, \ldots, x_n : \tau_n \vdash_{F\omega}$ iff $\Delta \vdash_{F\omega}$ and $\Delta; \emptyset \vdash_{HML} \tau_i$ for $1 \le i \le n$
  - $\Delta; \Gamma \vdash_{F\omega} e : \tau$ iff $\Delta; \Gamma \vdash_{F\omega}$, $\Delta; \emptyset \vdash_{HML} \tau$ and $\Delta; \Gamma \vdash_{HML} e : \tau$

- $\tilde{\ }: CC \to HML$ is restriction to split contexts, i.e.
  - $\Delta \vdash_{HML} u : k$ iff $\Delta \vdash_{CC} u : k$
  - $\Delta; \Gamma \vdash_{HML} e : \tau$ iff $\Delta, \Gamma \vdash_{CC} e : \tau$

- $\cdot : F\omega \to CC$ is the most complex and has to rely on a translation $\_^*$.
  - $\Phi \vdash_{CC}$ iff $\Phi^* \equiv \Delta; \Gamma$ and $\Delta; \Gamma \vdash_{F\omega}$
  - $\Phi \vdash_{CC} u : k$ iff $\Phi^* \equiv \Delta; \Gamma$, $k^* \equiv [v : k', \tau']$, $u^* \equiv [u', e']$ and $\Delta, \Gamma \vdash_{F\omega}$, $\Delta, v : k' \vdash_{F\omega} \tau'$, $\Delta \vdash_{F\omega} u' : k'$, $\Delta, \Gamma \vdash_{F\omega} e' : [u'/v]\tau'$
  - $\Phi \vdash_{CC} e : \tau$ iff $\Phi^* \equiv \Delta; \Gamma$, $\tau^* \equiv \tau'$, $e^* \equiv e'$ and $\Delta; \Gamma \vdash_{F\omega} e' : \tau'$

  the key clause in the definition of $\_^*$ (used in defining context extension) is $(\Phi, v : k)^* \equiv \Delta, v : k'; \Gamma, x_v : \tau'$, where $\Phi^* \equiv \Delta; \Gamma$ and $k^* \equiv [v : k', \tau']$.

3. Fibrations and Comprehension Categories.

Before starting with the precise mathematical exposition in 3.5 below, we describe the type-theoretic view on the notions defined in this section.

3.1. FIBRATIONS. A fibration $p : \mathbf{D} \to \mathbf{B}$ corresponds to a type theory with two levels, the second depending on the first, but with no other dependency. The judgements interpretable in such structure (and their interpretation) are:

judgement	interpretation
$x : B \vdash_p$	object B in the *base* $\mathbf{B}$
$x : B \vdash_p e(x) : B'$	morphism in the base
$x : B; y : D(x) \vdash_p$	object of *total category* $\mathbf{D}$
	"$;$" indicates the separation between the two levels
$x : B \vdash_p D(x)$	object in the *fibre* $\mathbf{D}_B$ over B
$x : B; y : D(x) \vdash_p [e(x), e'(x, y)] : [x' : B'; D'(x')]$	morphism in the total category
$x : B; y : D(x) \vdash_p e(x, y) : D'(x)$	morphism in the fibre over B

3.2. COMPREHENSION CATEGORIES. A comprehension category

$$\begin{tikzcd} \mathbf{D} \arrow[r, bend left, "\mathcal{P}_0"] \arrow[r, bend right, "p"'] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathbf{B} \end{tikzcd}$$

corresponds to a type theory with one level and type dependency (i.e. with two levels, but where the second can be *reflected* into the first). Such a diagram corresponds to a functor from $\mathbf{D}$ to $\mathbf{B}^{\to}$, the "arrow category" of $\mathbf{B}$. The kind of judgements interpretable in such structure are:

judgement	interpretation
$\Delta \vdash_{\mathcal{P}}$	object B in the base
$\Delta \vdash_{\mathcal{P}} D$	object D in the fibre over B
$\Delta \vdash_{\mathcal{P}} \phi : \Gamma$	morphism in the base
$\Delta \vdash_{\mathcal{P}} e : D$	*section* of $\mathcal{P}D$

$$\frac{\Delta \vdash_{\mathcal{P}} D}{\Delta, x : D \vdash_{\mathcal{P}}}$$

is the rule for context extension, and it corresponds to apply $\mathcal{P}_0$ to D. The arrow $\mathcal{P}D$ is the projection $\Delta, x : D \to \Delta$.

3.3. CHANGE-OF-BASE. If $\mathcal{P} : \mathbf{D} \to \mathbf{B}^{\to}$ and $q : \mathbf{E} \to \mathbf{B}$, then the comprehension category $q^*(\mathcal{P})$ over $\mathbf{E}$ is such that $\Delta; z : E \vdash_{q^*(\mathcal{P})} D$ iff $\Delta \vdash_q E$ and $\Delta \vdash_{\mathcal{P}} D$, while context extension is given by

$$\frac{\Delta; z : E \vdash_{q^*(\mathcal{P})} D}{\Delta, y : D; z : E \vdash_{q^*(\mathcal{P})}}$$

3.4. JUXTAPOSITION. If $\mathcal{P} : \mathbf{D} \to \mathbf{B}^{\to}$ and $\mathcal{Q} : \mathbf{E} \to \mathbf{B}^{\to}$, then the comprehension category $\mathcal{P} \cdot \mathcal{Q}$ over $\mathbf{B}$ is such that $\Delta \vdash_{\mathcal{P}\cdot\mathcal{Q}} [y : D, E]$ iff $\Delta \vdash_{\mathcal{P}} D$ and $\Delta, y : D \vdash_{\mathcal{Q}} E$, while context extension is given by

$$\frac{\Delta \vdash_{\mathcal{P}\cdot\mathcal{Q}} [y : D, E]}{\Delta, y : D, z : E \vdash_{\mathcal{P}\cdot\mathcal{Q}}}$$

3.5. FIBRED CATEGORY THEORY. Suppose we have a functor $p : \mathbf{E} \to \mathbf{B}$. An object $E \in \mathbf{E}$ (resp. a morphism f in $\mathbf{E}$) is said to be *above* $A \in \mathbf{B}$ (resp. u in $\mathbf{B}$) if $pE = A$ (resp. $pf = u$). A morphism above an identity is called *vertical*. Every object $A \in \mathbf{B}$ thus determines a so-called “fibre” category $\mathbf{E}_A$ consisting of objects above A and vertical morphisms. One often calls $\mathbf{B}$ the *base* category and $\mathbf{E}$ the *total* category.

A morphism $f : D \to E$ in $\mathbf{E}$ is called *cartesian* over a morphism u in $\mathbf{B}$ if f is above u and every $f' : D' \to E$ with $pf' = u \circ v$ in $\mathbf{B}$, uniquely determines a $\phi : D' \to D$ above v with $f \circ \phi = f'$. The functor $p : \mathbf{E} \to \mathbf{B}$ is called a *fibration* (sometimes a *fibred category* or a *category over* $\mathbf{B}$) if for every $E \in \mathbf{E}$ and $u : A \to pE$ in $\mathbf{B}$, there is a cartesian morphism with codomain E above u. Dually, $f : D \to E$ is *cocartesian* over u if every $f' : D \to E'$ with $pf' = v \circ u$, uniquely determines a $\phi : E \to E'$ above v with $\phi \circ f = f'$. And: p is a *cofibration* if every morphism $pE \to A$ in $\mathbf{B}$ has a “cocartesian lifting” with domain E. In case $\mathbf{B}$ is a category with pullbacks, the functor $cod : \mathbf{B}^{\to} \to \mathbf{B}$ forms an example of a fibration; it is at the same time a cofibration.

If $f : D \to E$ and $f' : D' \to E$ are both cartesian over u, then $f \cong f'$ in $\mathbf{E}/E$ by a vertical isomorphism. Hence given $u : A \to B$ in $\mathbf{B}$ and E above B, it makes sense to *choose* a cartesian lifting of u with codomain E; we often write $\overline{u}(E) : u^*(E) \to E$ for such a choice. Making similar choices for every $E \in \mathbf{E}_B$ determines a functor $u^* : \mathbf{E}_B \to \mathbf{E}_A$, called *inverse image*, *reindexing* or *substitution* functor. Such functors u^* are determined (by choice) up to vertical natural isomorphism. In general, one only has vertical natural isomorphisms $(u \circ v)^* \cong v^* \circ u^*$ and $id^* \cong Id$, as for pullbacks in case of $cod : \mathbf{B}^{\to} \to \mathbf{B}$. A fibration is *split* if it is given together with a choice of inverse images for which these isomorphisms are identities. Often, we suppose that a fibration comes equipped with a *cleavage*, i.e. an arbitrary choice of inverse images.

A morphism between fibrations p and q is given by a commuting square as below, in which the functor H preserves cartesian morphisms, i.e. f is p-cartesian implies that Hf is q-cartesian (such a functor is called *cartesian*).

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "H"'] \arrow[d, "p"'] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, "K"] & \mathbf{A}
\end{tikzcd}$$

This determines a (very large) "category" Fib. Given a fibration $q:\mathbf{D} \to \mathbf{A}$ and an arbitrary functor $K:\mathbf{B} \to \mathbf{A}$ one can form the pullback

$$\begin{tikzcd}
\mathbf{B} \underset{K,q}{\times} \mathbf{D} \arrow[r] \arrow[d, "K^*(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{D} \arrow[d, "q"] \\
\mathbf{B} \arrow[r, "K"] & \mathbf{A}
\end{tikzcd}$$

and verify that $K^*(q)$ is a fibration again. Consequently, the "functor" $Fib \longrightarrow \mathbf{Cat}$, sending a fibration to its base, is a fibration itself. Usually, one writes $Fib(\mathbf{B})$ for the "fibre" category of fibrations with base $\mathbf{B}$. The above construction is called *change-of-base* (for fibrations). $Fib(\mathbf{B})$ is in fact a 2-category with *vertical* natural transformations as 2-cells. Also Fib is a 2-category.

The "fibred" way of doing category theory over a base category was started by A. Grothendieck and further developed notably by J. Bénabou. For example, a *fibred* adjunction between fibrations $p:\mathbf{E} \to \mathbf{B}$ and $q:\mathbf{D} \to \mathbf{B}$ is given by a pair of cartesian functors $F:\mathbf{E} \to \mathbf{D}$ and $G:\mathbf{D} \to \mathbf{E}$ forming an adjunction $F \dashv G$ with vertical units and counits. These data determine adjunctions between the fibre categories, which are preserved under reindexing. Similarly, one says that a fibration has *fibred* cartesian products, exponents etc. if such a structure exists in the fibre categories and is preserved under reindexing, see Jacobs [1990] for a (more precise) description in terms of fibred adjunctions. Thus, fibred terminal objects for a fibration $p:\mathbf{E} \to \mathbf{B}$ may be described by a functor $1:\mathbf{B} \to \mathbf{E}$ such that $p \circ 1 = Id$ with the property that $1A$ is terminal in $\mathbf{E}_A$ and for $u:A \to B$ in $\mathbf{B}$ one has $u^*(1B) \cong 1A$.

There is one further notion that should be explained at this point. A fibration $p:\mathbf{E} \to \mathbf{B}$ is said to have a *generic object* if there is an object $T \in \mathbf{E}$ such that for every $E \in \mathbf{E}$ one can find a cartesian arrow $E \to T$

3.6. An elementary construction. Suppose a fibration $p:\mathbf{E} \to \mathbf{B}$ is given which has fibred finite products. A new fibration $\overline{p}:\overline{\mathbf{E}} \to \mathbf{E}$ is constructed in the following way. The category $\overline{\mathbf{E}}$ has pairs $E, E' \in \mathbf{E}$ with $pE = pE'$ as objects; morphisms $(f, g):(E, E') \to (D, D')$ in $\overline{\mathbf{E}}$ are given by arrows $f:E \to D$ and $g:E \times E' \to D'$ in $\mathbf{E}$ with $pf = pg$. The first projection $\overline{p}:\overline{\mathbf{E}} \to \mathbf{E}$ is then a fibration. One easily verifies that $\overline{p}$ has fibred finite products again and that it has a generic object in case p has one. Moreover that there is a change-of-base situation,

$$\begin{tikzcd}
\mathbf{E} \arrow[r, "H"'] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{\mathbf{E}} \arrow[d, "\overline{p}"] \\
\mathbf{B} \arrow[r, "1"] & \mathbf{E}
\end{tikzcd}$$

in which both 1 (for terminals) and H are full and faithful functors.

3.7. Definition. (Jacobs [1990]) A *comprehension category* is a functor $\mathcal{P}:\mathbf{E} \to \mathbf{B}^{\to}$ satisfying

(i) $cod \circ \mathcal{P} : \mathbf{E} \to \mathbf{B}$ is a fibration;

(ii) f cartesian in $\mathbf{E} \Rightarrow \mathcal{P}f$ is a pullback in $\mathbf{B}$.

This $\mathcal{P}$ is called a *full* comprehension category in case $\mathcal{P}$ is a full and faithful functor, and it is called a *cartesian* comprehension category in case every morphisms in $\mathbf{E}$ is cartesian.

3.8. Remarks. (i) For a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ we use the following standard notation: $p = cod \circ \mathcal{P}$ and $\mathcal{P}_0 = dom \circ \mathcal{P}$. The object part of $\mathcal{P}$ then forms a natural transformation $\mathcal{P} : \mathcal{P}_0 \stackrel{\cdot}{\longrightarrow} p$. (Similarly, for e.g. $\mathcal{Q} : \mathbf{D} \to \mathbf{A}^{\to}$, we write $q = cod \circ \mathcal{Q}$ and $\mathcal{Q}_0 = dom \circ \mathcal{Q}$.) The components $\mathcal{P}E$ are often called *projections* and reindexing functors of the form $\mathcal{P}E^*$ are called *weakening* functors.

(ii) If $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ is a comprehension category, one has that every object $E \in \mathbf{E}$ determines a pullback functor $\mathcal{P}E^{\#} : \mathbf{B}/pE \to \mathbf{B}/\mathcal{P}_0 E$ by $u \mapsto \mathcal{P}_0(\overline{u}(E))$.

(iii) In Ehrhard [1988] a *D-category* is defined as a fibration $p : \mathbf{E} \to \mathbf{B}$ provided with a terminal object functor $1 : \mathbf{B} \to \mathbf{E}$, which has a right adjoint $\mathcal{P}_0 : \mathbf{E} \to \mathbf{B}$. The ensuing functor $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ given by $E \mapsto p(\varepsilon_E)$ then forms a comprehension category (where $\varepsilon : 1\mathcal{P}_0 \stackrel{\cdot}{\longrightarrow} Id$ is counit). Two things are worth noticing.

(a) This functor $\mathcal{P}$ preserves the terminal objects, i.e. for $A \in \mathbf{B}$, the map $\mathcal{P}1A$ is an isomorphism (i.e. terminal in $\mathbf{B}/A$). Since 1 is a full and faithful functor, the unit $\eta : Id \stackrel{\cdot}{\longrightarrow} \mathcal{P}_0 1$ is an iso. But $\mathcal{P}1 \circ \eta = p\varepsilon 1 \circ \eta = p\varepsilon 1 \circ p1\eta = p(\varepsilon 1 \circ 1\eta) = id$, which makes $\mathcal{P}1$ an iso as well.

(b) For $E \in \mathbf{E}$ and $u : A \to pE$ in $\mathbf{B}$ one has

$$\mathbf{B}/pE\,(u,\ \mathcal{P}E) \;\cong\; \mathbf{E}_A\,(1A,\ u^*(E)),$$

which can be verified by playing a bit with the adjunction $1 \dashv \mathcal{P}_0$. We understand D-categories as forming a suitable concept of "comprehension categories with a unit". Therefore, we'll say that a comprehension category has (or admits) a unit if it is a D-category. This renaming gives more uniformity.

The essential point about a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ is that it determines a class $\{\mathcal{P}E \mid E \in \mathbf{E}\}$ of "display" maps in $\mathbf{B}$, which behave well in a certain sense. The abstract formulation of comprehension categories has technical and methodological advantages. This section concludes with two constructions on comprehension categories; the first one is from Jacobs [1991] and the second one from Moggi [1991].

3.9. Definition (Change-of-base for comprehension categories along fibrations). Given a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ and a fibration $q : \mathbf{D} \to \mathbf{B}$, a new comprehension category $q^*(\mathcal{P})$ with base category $\mathbf{D}$ can be constructed as follows. First form the fibration $q^*(p)$ by change-of-base

$$\begin{tikzcd}
\mathbf{D} \underset{q,p}{\times} \mathbf{E} \arrow[r] \arrow[d, "q^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{D} \arrow[r, "q"] & \mathbf{B}
\end{tikzcd}$$

and then choose $q^*(\mathcal{P}) : \mathbf{D} \underset{q,p}{\times} \mathbf{E} \to \mathbf{D}^{\to}$ by $(D, E) \mapsto \overline{\mathcal{P}E}(D) : \mathcal{P}E^*(D) \to D$. On arrows $(f, g) : (D, E) \to (D', E')$ where $qf = pg$ one defines $q^*(\mathcal{P})(f, g) = (f, h)$, in which

$h : \mathcal{P}E^*(D) \to \mathcal{P}E'^*(D')$ is the unique arrow above $\mathcal{P}_0 g$ satisfying $\overline{\mathcal{P}E'}(D') \circ h = f \circ \overline{\mathcal{P}E}(D)$.

3.10. Definition (Juxtaposition of comprehension categories). Starting from two comprehension categories $\mathbf{E} \xrightarrow{\mathcal{P}} \mathbf{B}^{\to} \xleftarrow{\mathcal{Q}} \mathbf{D}$ one constructs another comprehension category $\mathcal{Q} \cdot \mathcal{P}$ with base category $\mathbf{B}$, by first performing change-of-base

$$\begin{tikzcd} \mathbf{D} \underset{\mathcal{Q}_0, p}{\times} \mathbf{E} \arrow[r] \arrow[d, "\mathcal{Q}_0^*(p)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\ \mathbf{D} \arrow[r, "\mathcal{Q}_0"] & \mathbf{B} \end{tikzcd}$$

and then defining $\mathcal{Q} \cdot \mathcal{P} : \mathbf{D} \underset{\mathcal{Q}_0, p}{\times} \mathbf{E} \to \mathbf{B}^{\to}$ by $(D, E) \mapsto \mathcal{Q}D \circ \mathcal{P}E$ and $(f, g) \mapsto (qf, \mathcal{P}_0 g)$. One has $cod \circ \mathcal{Q} \cdot \mathcal{P} = q \circ \mathcal{Q}_0^*(p)$.

3.11. Lemma. (i) $\mathcal{P}$ is full $\Rightarrow$ $q^*(\mathcal{P})$ is full;
$\mathcal{P}$ has a unit $\Rightarrow$ $q^*(\mathcal{P})$ has a unit.

(ii) $\mathcal{P}, \mathcal{Q}$ have units $\Rightarrow$ $\mathcal{Q} \cdot \mathcal{P}$ has a unit. Moreover, there is a full and faithful functor $\mathcal{I} : cod \circ \mathcal{P} \to cod \circ \mathcal{Q} \cdot \mathcal{P}$ preserving the terminal object and satisfying $\mathcal{Q} \cdot \mathcal{P} \circ \mathcal{I} \cong \mathcal{P}$. $\Box$

4. Fibred Products and Sums.

Comprehension categories will be used in two different, but related ways: in this section to define appropriate fibred notions of product and sum; in the sixth section to provide categories for type dependency.

A comprehension category determines a class of "projection" morphisms. Quantification along such projections is described in the next definition by adjoints to the corresponding "weakening" functors. We first mention that a fibration $p : \mathbf{E} \to \mathbf{B}$ determines a category $Cart(\mathbf{E})$ with all objects from $\mathbf{E}$, but only the cartesian arrows. By restriction one obtains a fibration $|p| : Cart(\mathbf{E}) \to \mathbf{B}$. Similarly a comprehension category $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ determines two functors $|p|$, $|\mathcal{P}_0| : Cart(\mathbf{E}) \to \mathbf{B}$ and a natural transformation between them. Hence one obtains a cartesian comprehension category $|\mathcal{P}| : Cart(\mathbf{E}) \to \mathbf{B}^{\to}$.

4.1. Definition. Let $q : \mathbf{D} \to \mathbf{B}$ be a fibration and $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ a comprehension category. By change-of-base of q along the above functors $|p|$ and $|\mathcal{P}_0|$ one obtains two fibrations $|p|^*(q)$ and $|\mathcal{P}_0|^*(q)$. There is a cartesian functor $\langle\mathcal{P}\rangle : |p|^*(q) \to |\mathcal{P}_0|^*(q)$ described by $(E, D) \mapsto (E, \mathcal{P}E^*(D))$. We say that q *admits $\mathcal{P}$-products* (resp. *$\mathcal{P}$-sums*) if this functor $\langle\mathcal{P}\rangle$ has a fibred right (resp. left) adjoint.

4.2. Remarks. (i) The definition of the cartesian functor $\langle\mathcal{P}\rangle$ is based on proposition 3 in Ehrhard's [1988]. In fact this whole definition is inspired by his approach.

(ii) The above fibred definition has a more practical equivalent: let q and $\mathcal{P}$ be as above; then q admits $\mathcal{P}$-products (resp. $\mathcal{P}$-sums) iff both

- for every $E \in \mathbf{E}$, every weakening functor $\mathcal{P}E^* : \mathbf{D}_{pE} \to \mathbf{D}_{\mathcal{P}_0 E}$ has a right adjoint Π_E (resp. a left adjoint Σ_E).

- the "Beck-Chevalley" condition holds, i.e. for every cartesian morphism $f : E \to E'$ in $\mathbf{E}$ one has that the canonical natural transformation

$$(pf)^*\Pi_{E'} \stackrel{\cdot}{\longrightarrow} \Pi_E(\mathcal{P}_0 f)^* \qquad\qquad (\text{resp. } \Sigma_E(\mathcal{P}_0 f)^* \stackrel{\cdot}{\longrightarrow} (pf)^*\Sigma_{E'}\ )$$

  is an isomorphism.

The first map is the transpose of $\mathcal{P}E^*\,(pf)^*\,\Pi_{E'} \cong (\mathcal{P}_0 f)^*\,\mathcal{P}E'^*\,\Pi_{E'} \xrightarrow{\varepsilon} (\mathcal{P}_0 f)^*$; similarly one obtains the second one. In the sequel, we'll use products and sums in this "fibrewise" form.

(iii) A more economical approach would be to define $\mathcal{P}$-quantification for full cartesian comprehension categories $\mathcal{P}$ only. However, as will become clear later, the extra generality we have in the above definition has advantages.

4.3. Lemma. Suppose q admits $\mathcal{P}$-sums as defined above. For every $E \in \mathbf{E}$ and $D \in \mathbf{D}$ with $qD = \mathcal{P}_0 E$, one has that the morphism $in_{E,D} = \overline{\mathcal{P}E}(\Sigma_E.D) \circ \eta_D : D \to \mathcal{P}E^*(\Sigma_E.D) \to \Sigma_E.D$ is cocartesian. 2

In type theory one finds so-called "weak" and "strong" sums, see section 6. The above definition covers the weak case. For the strong one q must be (part of) a comprehension category.

4.4. Definition. Given comprehension categories $\mathbf{E} \xrightarrow{\mathcal{P}} \mathbf{B}^{\to} \xleftarrow{\mathcal{Q}} \mathbf{D}$, we say that

(i) $\mathcal{Q}$ has $\mathcal{P}$-products/sums in case $q = cod \circ \mathcal{Q}$ has $\mathcal{P}$-products/sums.

(ii) $\mathcal{Q}$ has *strong* $\mathcal{P}$-sums in case $\mathcal{Q}$ has $\mathcal{P}$-sums in such a way that every morphism $\mathcal{Q}_0(in_{E,D})$ in $\mathbf{B}$ (cf. the previous lemma) is orthogonal to the class $\{\mathcal{Q}D' \mid D' \in \mathbf{D}\}$.

The latter means that for every $D' \in \mathbf{D}$ and u, v forming a commuting square,

$$\begin{tikzcd}
\bullet \arrow[r, "\mathcal{Q}_0(in_{E,D})"] \arrow[d, "u"'] & \bullet \arrow[d, "v"] \arrow[dl, "w"] \\
\bullet \arrow[r, "\mathcal{Q}D'"'] & \bullet
\end{tikzcd}$$

there is a unique w satisfying $\mathcal{Q}D' \circ w = v$ and $w \circ \mathcal{Q}_0(in_{E,D}) = u$.

We say that a comprehension category $\mathcal{Q}$ admits products/(strong) sums if it admits $\mathcal{Q}$-products/(strong) sums. One easily verifies that $\mathcal{Q}$ has strong sums iff the above morphism $\mathcal{Q}_0(in_{E,D})$ is an isomorphism. The latter formulation is used in Jacobs [1990] to define strong sums for comprehension categories.

The notion to be introduced next is of great importance — see e.g. the subsequent examples. It provides a suitable "unit" or "building-block" to describe more complicated categories later.

4.5. Definition. A *closed* comprehension category (abbr. CCompC) is a full comprehension category with unit, products and strong sums; moreover, the base category is required to have a terminal object.

4.6. Examples. (i) Let $\mathbf{B}$ be a category with finite limits. The identity functor on $\mathbf{B}^{\rightarrow}$ then forms a full comprehension category with unit and strong sums. Moreover,

$$id_{\mathbf{B}^{\rightarrow}} \text{ is a CCompC } \Leftrightarrow \mathbf{B} \text{ is a LCCC.}$$

(ii) Let $\mathbf{B}$ be a category with finite products. The functor $\mathbf{B} \rightarrow \mathbf{1}$ (the terminal category) then forms a fibration with finite products. Hence the construction from 3.6 yields a fibration $\overline{\mathbf{B}} \rightarrow \mathbf{B}$. The functor $Cons_{\mathbf{B}} : \overline{\mathbf{B}} \rightarrow \mathbf{B}^{\rightarrow}$ given by $(A, A') \mapsto [\pi : A \times A' \rightarrow A]$ forms a full comprehension category with unit and strong sums. Moreover,

$$Cons_{\mathbf{B}} \text{ is a CCompC } \Leftrightarrow \mathbf{B} \text{ is a CCC.}$$

The rest of this section will be devoted to technical results about products and sums and closed comprehension categories.

4.7. Lemma. Let $\mathcal{P} : \mathbf{E} \rightarrow \mathbf{B}^{\rightarrow}$ be a comprehension category and $q : \mathbf{D} \rightarrow \mathbf{B}$ a fibration.

(i) A fibration r admits $\mathcal{P}$-products/sums $\Rightarrow$ $q^*(r)$ admits $q^*(\mathcal{P})$-products/ sums.

(ii) A comprehension category $\mathcal{R}$ admits strong $\mathcal{P}$-sums $\Rightarrow$ $q^*(\mathcal{R})$ admits strong $q^*(\mathcal{P})$-sums.

Proof. Suppose the fibration r has $\mathbf{C}$ as total category, i.e. one has $r : \mathbf{C} \rightarrow \mathbf{B}$.

(i) Assume $\Sigma_E \dashv \mathcal{P}E^*$ in $\mathbf{C}$; we have to construct $\exists_{(D,E)} \dashv (q^*(\mathcal{P})(D,E))^*$. This is done by defining $\exists_{(D,E)} : (\mathbf{D} \underset{q,r}{\times} \mathbf{C})_{\mathcal{P}E^*(D)} \rightarrow (\mathbf{D} \underset{q,r}{\times} \mathbf{C})_D$ as $(\mathcal{P}E^*(D), C) \mapsto (D, \Sigma_E.C)$. Products are handled similarly.

(ii) Notice that one has $in = in_{(D,E),(\mathcal{P}E^*(D),C)} = (\overline{\mathcal{P}E}(D), in_{E,C}) : (\mathcal{P}E^*(D), C) \rightarrow \exists_{(D,E)}.(\mathcal{P}E^*(D), C)$ and that $q^*(\mathcal{P})_0(in)$ is by definition above $\mathcal{R}_0(in_{E,C})$. Orthogonilty can then be lifted. $\Box$

4.8. Lemma. Let $\mathbf{E} \xrightarrow{\mathcal{P}} \mathbf{B}^{\rightarrow} \xleftarrow{\mathcal{Q}} \mathbf{D}$ be comprehension categories.

(i) A fibration r admits both $\mathcal{P}$- and $\mathcal{Q}$-products/sums $\Rightarrow$ r admits $\mathcal{Q}{\cdot}\mathcal{P}$-products/sums.

(ii) A comprehension category $\mathcal{R}$ admits both strong $\mathcal{P}$- and strong $\mathcal{Q}$-sums $\Rightarrow$ $\mathcal{R}$ admits strong $\mathcal{Q} \cdot \mathcal{P}$-sums.

Proof. (i) By composition of adjoints.

(ii) By successive application of orthogonality. $\Box$

4.9. Lemma. Consider two comprehension categories $\mathbf{E} \xrightarrow{\mathcal{P}} \mathbf{B}^{\rightarrow} \xleftarrow{\mathcal{Q}} \mathbf{D}$.

(i) $\mathcal{Q}$ has strong sums $\Rightarrow$ $\mathcal{Q} \cdot \mathcal{P}$ has strong $\mathcal{Q}$-sums.

(ii) In case $\mathcal{P}$ has a unit,

$\mathcal{P}$ and $\mathcal{Q}$ both have $\mathcal{Q}$-products $\Rightarrow$ $\mathcal{Q} \cdot \mathcal{P}$ has $\mathcal{Q}$-products.

Proof. (i) Let $\Sigma_D \dashv \mathcal{Q}D^*$ in $\mathbf{D}$ be given; we have to construct $\exists_D \dashv \mathcal{Q}D^*$ in $\mathbf{D} \underset{\mathcal{Q}_0,p}{\times} \mathbf{E}$. This is done by taking for $(D_1, E_1) \in (\mathbf{D} \underset{\mathcal{Q}_0,p}{\times} \mathbf{E})$ above $\mathcal{Q}_0 D$,

$$\exists_D.(D_1, E_1) = (\Sigma_D.D_1,\ \mathcal{Q}_0(in_{D,D_1})^{-1\,*}(E_1)).$$

(ii) We may assume adjunctions $\mathcal{Q}D^* \dashv \Pi_D$ in $\mathbf{D}$ and $\mathcal{Q}D^* \dashv \forall_D$ in $\mathbf{E}$. One takes

$$\forall_D.(D_1, E_1) = (D',\ \forall_{\mathcal{Q}D'^*(D)}.\ \phi^*\ \mathcal{Q}_0(\varepsilon_{D_1})^*(E_1)\,),$$

where $D' = \Pi_D.D_1$ and $\varepsilon_{D_1} : \mathcal{Q}D^*(\Pi_D.D_1) \to D_1$ is unit and ϕ is an obvious mediating isomorphism in $\mathbf{B}$. In order to obtain the required adjunction, one has to use that $\mathcal{P}$ preserves products, see the proof of 4.11 (ii). $\Box$

4.10. Lemma. Let $q : \mathbf{D} \to \mathbf{B}$ be a fibration and $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category.

(i) If there is a fibred reflection $r \to q$ (i.e. a fibration $r : \mathbf{C} \to \mathbf{B}$ and a full and faithful cartesian functor $\mathbf{C} \to \mathbf{D}$ which has a fibred left adjoint), then

$$q \text{ has } \mathcal{P}\text{-products/sums} \quad \Rightarrow \quad r \text{ has } \mathcal{P}\text{-products/sums.}$$

Moreover, the functor $\mathbf{C} \to \mathbf{D}$ preserves the products.

(ii) In case $\mathcal{P}$ is a full comprehension category with unit and sums and q has a fibred terminal object, which is preserved by a full and faithful cartesian functor $G : \mathbf{D} \to \mathbf{E}$, then

$$G \text{ has a fibred left adjoint} \quad \Leftrightarrow \quad q \text{ has } \mathcal{P}\text{-sums.}$$

Proof. (i) Standard.

(ii) ($\Rightarrow$) By (i).

($\Leftarrow$) Define $F : \mathbf{E} \to \mathbf{D}$ by $E \mapsto \Sigma_E(\top\mathcal{P}_0 E)$, where $\top : \mathbf{B} \to \mathbf{D}$ describes the terminal object for q. By 4.3, F extends to a functor, which is cartesian by Beck-chevalley. $\Box$

4.11. Lemma. Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a CCompC. Then

(i) $p = cod \circ \mathcal{P} : \mathbf{E} \to \mathbf{B}$ is a fibred CCC;

(ii) Considered as a functor, $\mathcal{P}$ preserves units, sums and products.

Proof. (i) One takes $E \times E' = \Sigma_E.\mathcal{P}E^*(E')$ and $E \Rightarrow E' = \Pi_E.\mathcal{P}E^*(E')$. In fact, strongness of the sums is not needed to obtain this, see Jacobs [1990], 5.2.3.

(ii) Units are preserved by remark 3.8 (iii) (a) and sums by strongness: $\mathcal{P}(\Sigma_E.E') \cong \mathcal{P}E \circ \mathcal{P}E' = \Sigma_{\mathcal{P}E}.\mathcal{P}E'$ in $\mathbf{B}/pE$. As to products we obtain for $u : A \to pE$ in $\mathbf{B}$,

$$\begin{array}{rcl}
\mathbf{B}/pE\ (u,\ \mathcal{P}(\Pi_E.E')\,) & \cong & \mathbf{E}_A\ (1A,\ u^*(\Pi_E.E')\,) \\
& \cong & \mathbf{E}_A\ (1A,\ \Pi_{u^*(E)}.(\mathcal{P}E^{\#}(u))^*(E')\,), \text{ by Beck-Chevalley} \\
& \cong & \mathbf{E}_{\mathcal{P}_0 u^*(E)}\ (\,(\mathcal{P}u^*(E))^*(1A),\ (\mathcal{P}E^{\#}(u))^*(E')\,) \\
& \cong & \mathbf{E}_{\mathcal{P}_0 u^*(E)}\ (\,1\mathcal{P}u^*(E),\ (\mathcal{P}E^{\#}(u))^*(E')\,) \\
& \cong & \mathbf{B}/\mathcal{P}_0 E\ (\mathcal{P}E^{\#}(u),\ \mathcal{P}E')
\end{array}$$

in which the pullback functor $\mathcal{P}E^{\#}$ comes from 3.8 (ii). The first and last step hold by 3.8 (iii) (b) $\Box$

4.12. Proposition. $\mathcal{P}$ is a CCompC $\Rightarrow$ $q^*(\mathcal{P})$ is a CCompC.

Proof. Let $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ be a full comprehension category with unit, products and strong sums. $q^*(\mathcal{P})$ is again full and has a unit by 3.11 (ii); it admits products and strong sums by 4.7. $\Box$

5. Category Theory over a Fibration

In the first section it was explained how a fibration forms a category fibred over a base category. Now we go one step up and consider categories over a fibration. This is not as bad as it may seem, since it turns out that one can reduce matters to the previous level. The following lemma lies at the basis of all this.

5.1. Lemma. Let $p : \mathbf{E} \to \mathbf{B}$ and $r : \mathbf{B} \to \mathbf{A}$ be fibrations.

(i) The functor $rp : \mathbf{E} \to \mathbf{A}$ is a fibration, with

$$f \text{ is } rp\text{-cartesian} \quad \Leftrightarrow \quad f \text{ is } p\text{-cartesian and } pf \text{ is } r\text{-cartesian.}$$

(ii) The functor p is cartesian from rp to r.

(iii) If $q : \mathbf{D} \to \mathbf{B}$ is another fibration, then

$$F : p \to q \text{ in } Fib(\mathbf{B}) \quad \Rightarrow \quad F : rp \to rq \text{ in } Fib(\mathbf{A}). \quad \square$$

5.2. A fibration as a basis. Suppose a cartesian functor p is given as in the following diagram.

$$\begin{tikzcd} \mathbf{E} \arrow[rr, "p"'] \arrow[dr, "q"'] & & \mathbf{B} \arrow[dl, "r"] \\ & \mathbf{A} & \end{tikzcd}$$

Every object $A \in \mathbf{A}$ determines a "fibrewise" functor $p|_A : \mathbf{E}_A \to \mathbf{B}_A$ by restriction. One calls p a *fibration over r* if all these fibrewise functors are fibrations and reindexing functors preserve the relevant cartesian structure (similarly to the definition of e.g. fibred cartesian products). More explicitly, p is a fibration over r if both

- for very $A \in \mathbf{A}$, $p|_A$ is a fibration;
- for every $u : A \to A'$ in $\mathbf{A}$ and every r-reindexing functor $u^* : \mathbf{B}_{A'} \to \mathbf{B}_A$, there is a q-reindexing functor $u^\# : \mathbf{E}_{A'} \to \mathbf{E}_A$ forming a morphism of fibrations:

$$\begin{tikzcd} \mathbf{E}_{A'} \arrow[r, "u^\#"'] \arrow[d, "p|_{A'}"'] & \mathbf{E}_A \arrow[d, "p|_A"] \\ \mathbf{B}_{A'} \arrow[r, "u^*"] & \mathbf{B}_A \end{tikzcd}$$

This rather complicated notion is equivalent to a more simple one; namely

$$p \text{ is a fibration over } r \;\Leftrightarrow\; p \text{ is a fibration itself.}$$

To verify the implication $(\Leftarrow)$, notice that $p|_A$ can be obtained from p by change-of-base. This yields that f in $\mathbf{E}_A$ is $p|_A$-cartesian iff f is p-cartesian. The rest is not difficult. As to the implication $(\Rightarrow)$, observe that if p is a fibration over r, then f in $\mathbf{E}$ is p-cartesian iff f can be written as $g \circ \alpha$ where g is q-cartesian and α is $p|_A$-cartesian (with $A = q(dom\ f)$).

Next, suppose we have a diagram,

$$
\begin{tikzcd}
\mathbf{E} \arrow[rr, "F"'] \arrow[dr, "p"] \arrow[ddr, "q"'] & & \mathbf{E}' \arrow[dl, "p'"'] \arrow[ddl, "q'"] \\
& \mathbf{B} \arrow[d, "r"] & \\
& \mathbf{A} &
\end{tikzcd}
$$

in which r, q, q', p and p' are fibrations with $q = rp$, $q' = rp'$ and F is a cartesian functor from q to q'. One calls F a *cartesian functor from p to p' over r* if both

- $p' \circ F = p$.
- for every $A \in \mathbf{A}$, $F|_A$ is cartesian form $p|_A$ to $p|_{A'}$;

As before, one can show that

$$F \text{ is cartesian } p \to p' \text{ over } r \;\Leftrightarrow\; F \text{ is cartesian } p \to p' \text{ in } \mathit{Fib}(\mathbf{B}).$$

In this way, one obtains a category $\mathit{Fib}(r)$ of fibrations and cartesian functors over r. As shown, one has $\mathit{Fib}(r) = \mathit{Fib}(\mathbf{B})$. It is left to the reader to formulate what natural transformations over r are and that the previous identification also concerns the 2-structure. Hence adjunctions over $r : \mathbf{B} \to \mathbf{A}$ are adjunctions over $\mathbf{B}$ (i.e. in the 2-category $\mathit{Fib}(\mathbf{B})$). In order to get an even better picture, the reader may want to verify that for $F : p \to p'$ in $\mathit{Fib}(\mathbf{B})$ as above and $G : p' \to p$ one has that $F \dashv G$ is an adjunction over r iff both

- for every $A \in \mathbf{A}$, there is fibred adjunction $F|_A \dashv G|_A$ in $\mathit{Fib}(\mathbf{B}_A)$;
- for every morphism $A \to A'$ in $\mathbf{A}$ there is a morphism $F|_{A'} \dashv G|_{A'} \longrightarrow F|_A \dashv G|_A$ of fibred adjunctions (see Jacobs [1990], 2.2 for the definition).

The above exposition is based on work of J. Bénabou; see also Pavlović [1990].

5.3. Definition. Let $r : \mathbf{B} \to \mathbf{A}$ be a fibration. A functor $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ is a *comprehension category over r* if $\mathcal{P}$ is a comprehension category in such a way that the functor $\mathcal{P}_0 = \mathit{dom} \circ \mathcal{P}$ is cartesian in

$$
\begin{tikzcd}
\mathbf{E} \arrow[rr, "\mathcal{P}_0"'] \arrow[dr, "rp"'] & & \mathbf{B} \arrow[dl, "r"] \\
& \mathbf{A} &
\end{tikzcd}
$$

and $\mathcal{P}$ has r-vertical components.

5.4. Remarks. (i) It takes a bit of effort, but one can show that given fibrations $\mathbf{E} \xrightarrow{p} \mathbf{B} \xrightarrow{r} \mathbf{A}$ and a 2-cell $\mathcal{P}\colon \mathcal{P}_0 \stackrel{\cdot}{\longrightarrow} p\colon rp \to r$ in $Fib(\mathbf{A})$, one has that $\mathcal{P}$ is a comprehension category over r iff for every $A \in \mathbf{A}$ one has a comprehension category $\mathcal{P}|_A\colon \mathbf{E}_A \to \mathbf{B}_A^{\to}$ and reindexing functors form suitable maps between these. Moreover, $\mathcal{P}$ is a full comprehension category iff all $\mathcal{P}|_A$'s are full, see Jacobs [1991] for more details.

(ii) We mention two examples.

(a) Let $\mathbf{B}$ be a category with pullbacks. The (obvious) functor $cod^{\to}\colon \mathbf{B}^{\to\to} \to \mathbf{B}^{\to}$ forms a fibration over $cod\colon \mathbf{B}^{\to} \to \mathbf{B}$. One obtains a full comprehension category $\mathbf{B}^{\to\to} \longrightarrow \mathbf{B}^{\to\to}$ over cod by $[\xrightarrow{v}\xrightarrow{u}] \mapsto [(id, v)\colon u \circ v \to u]$ in $\mathbf{B}^{\to}$.

(b) Let $p\colon \mathbf{E} \to \mathbf{B}$ be a fibration with finite products as in 3.6. One defines functor $\overline{\mathcal{P}}\colon \overline{\mathbf{E}} \to \mathbf{E}^{\to}$ by $(E, E') \mapsto \pi\colon E \times E' \to E$, which forms a full comprehension category over p. This generalizes $Cons_{\mathbf{B}}\colon \overline{\mathbf{B}} \to \mathbf{B}^{\to}$ from 4.6.

5.5. Definition. Let $p\colon \mathbf{E} \to \mathbf{B}$ and $r\colon \mathbf{B} \to \mathbf{A}$ be fibrations; p forms a *comprehension category with unit over* r if there is

- a terminal object functor $1\colon \mathbf{B} \to \mathbf{E}$ for p in $Fib(\mathbf{B})$;
- a fibred right adjoint $\mathcal{P}_0$ of $1\colon r \to rp$ in $Fib(\mathbf{A})$.

(Ordinary comprehension categories with unit are described in remark 3.8 (iii).)

5.6. Definition. A *closed comprehension category over a fibration* r is a full comprehension category with unit over r which admits products and strong sums (as a comprehension category in itself, not over r); moreover, r is required to have a fibred terminal object.

5.7. Examples. (i) If $\mathbf{B}$ is a LCCC, one obtains a CCompC over $cod\colon \mathbf{B}^{\to} \to \mathbf{B}$, see remark 5.4 (ii) (a). This comes from the fact that the slice categories $\mathbf{B}/A$ are LCCC's again using the same ingredients.

(ii) The other example mentioned in remark 5.4 (ii) is also of interest; it gives rise to a generalization of the equivalence obtained in 4.6 (ii). For a fibration $p\colon \mathbf{E} \to \mathbf{B}$ with finite products one has

$$\overline{\mathcal{P}}\colon \overline{\mathbf{E}} \to \mathbf{E}^{\to} \text{ is a CCompC over } p \quad \Leftrightarrow \quad p \text{ is a fibred CCC.}$$

In the next construction (from Jacobs [1991]), a generalization of $\overline{p}$ from 3.6 is obtained by using strong sums instead of cartesian products. In fact, the first example above is obtained in this way from $id_{\mathbf{B}^{\to}}$.

5.8. Proposition. Let $\mathcal{P}\colon \mathbf{E} \to \mathbf{B}^{\to}$ be a closed comprehension category. By change-of-base, we form the fibration $\tilde{p}\colon \tilde{\mathbf{E}} \to \mathbf{E}$.

$$\begin{tikzcd}
\tilde{\mathbf{E}} = \mathbf{E} \underset{\mathcal{P}_0, p}{\times} \mathbf{E} \arrow[rr] \arrow[d, "\tilde{p}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbf{E} \arrow[d, "p"] \\
\mathbf{E} \arrow[rr, "\mathcal{P}_0"] & & \mathbf{B}
\end{tikzcd}$$

Then

(i) $\tilde{p}: \widetilde{\mathbf{E}} \to \mathbf{E}$ forms part of a CCompC $\widetilde{\mathcal{P}}: \widetilde{\mathbf{E}} \to \mathbf{E}^{\to}$ over p;
(ii) there is a "pseudo" change-of-base situation (in which 1 is terminal object functor),

$$\begin{tikzcd} \mathbf{E} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\mathbf{E}} \arrow[d, "\tilde{p}"] \\ \mathbf{B} \arrow[r, "1"] & \mathbf{E} \end{tikzcd}$$

By "pseudo" we mean that the pullback of $\tilde{p}$ along 1 *yields a fibration which is equivalent instead of isomorphic to p (in the fibred sense).*

Proof. (i) One defines $\widetilde{\mathcal{P}}: \widetilde{\mathbf{E}} \to \mathbf{E}^{\to}$ by $(E, E') \mapsto$ [the projection $\Sigma_E.E' \to E$]; it is the unique map f with $\mathcal{P}_0 f = \mathcal{P}E' \circ \mathcal{P}_0(in_{E,E'})^{-1}$, using the morphism described in lemma 4.3 and the fact that $\mathcal{P}$ is full. The cocartesianess obtained in this lemma is needed to define $\widetilde{\mathcal{P}}$ on morphisms. The rest is laborious but straightforward.
(ii) Easy. $\Box$

The constructions $\overline{p}$ and $\tilde{p}$ provide two ways to obtain closed comprehension categories over a fibration. The next two lemmas state that quantification for the base fibration p can be lifted to $\overline{p}$ and $\tilde{p}$. One gets strongness of the lifted sums for free. Proofs are omitted, but may be found in Jacobs [1991].

5.9. Lemma. Let $p: \mathbf{E} \to \mathbf{B}$ be a fibred CCC and $\mathcal{Q}: \mathbf{D} \to \mathbf{B}^{\to}$ a comprehension category. Then

$$p \text{ admits } \mathcal{Q} \text{ products and sums} \quad \Rightarrow \quad \overline{\mathcal{P}} \text{ admits } p^*(\mathcal{Q}) \text{ products and strong sums.} \quad \Box$$

5.10. Lemma. Let $\mathcal{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a closed comprehension category and $\mathcal{Q}: \mathbf{D} \to \mathbf{B}^{\to}$ an arbitrary comprehension category. Then

$$p \text{ admits } \mathcal{Q}\text{-products and sums} \quad \Rightarrow \quad \widetilde{\mathcal{P}} \text{ admits } p^*(\mathcal{Q})\text{-products and strong sums.} \quad \Box$$

In order to obtain the result of the next lemma, Moggi [1991] used the auxiliary notion of *independence*. This can be avoided with comprehension categories *over a fibration*.

5.11. Lemma. Let $\mathcal{P}: \mathbf{E} \to \mathbf{B}^{\to}$ be a comprehension category over $r: \mathbf{B} \to \mathbf{A}$ and let $\mathcal{Q}: \mathbf{D} \to \mathbf{A}^{\to}$ be an ordinary comprehension category. We form $\mathcal{S} = r^*(\mathcal{Q}) \cdot \mathcal{P}$ with base category $\mathbf{B}$, see 3.9 and 3.10.
(i) $\mathcal{P}, \mathcal{Q}$ are full $\Rightarrow$ $\mathcal{S}$ is full.
(ii) $\mathcal{P}, \mathcal{Q}$ have a unit $\Rightarrow$ $\mathcal{S}$ has a unit.
(iii) $\mathcal{P}$ has products (resp. strong sums) $\Rightarrow$ $\mathcal{S}$ has $\mathcal{P}$-products (resp. strong sums). $\Box$

6. Models

In this section we describe categorical versions of the typed λ-calculi $F\omega$, the Calculus of Constructions(CC) and HML. The first system (due to J.-Y. Girard and J. Reynolds) and the second one (due to Th. Coquand and G. Huet) are considered to be well-known. The latter one comes from Moggi [1991] — see also Pavlović [1990] for a comparable system — and is essentially $F\omega$ extended with the possibility of types depending on types and propositions on propositions.

The next definition is based on Seely [1987].

6.1. DEFINITION. A *PL-category* is a fibred CCC $p : \mathbf{E} \to \mathbf{B}$ with a CCC $\mathbf{B}$ as basis, admitting a generic object and $Cons_{\mathbf{B}}$-products and sums.

For the Calculus of Construction, we give a fibred version of a notion from Hyland & Pitts [1989]. Later, we briefly mention a weaker version.

6.2. DEFINITION. A *CC-category* is described by the following data.

(i) A CCompC $\mathcal{Q} : \mathbf{D} \to \mathbf{B}^{\to}$.

(ii) A fibration $p : \mathbf{E} \to \mathbf{B}$ together with a fibred terminal object and a full and faithful cartesian functor $\mathcal{I} : \mathbf{E} \to \mathbf{D}$ which preserves this terminal. Further, we require that the comprehension category (with unit) $\mathcal{P} = \mathcal{QI} : \mathbf{E} \to \mathbf{B}^{\to}$ admits strong $\mathcal{Q}$-sums. In that case also $\mathcal{P}$ is a CCompC, see 4.10.

(iii) An object $\Omega \in \mathbf{D}$ such that $q\Omega \in \mathbf{B}$ is terminal and there is a generic object for p in $\mathbf{E}$ above $\mathcal{Q}_0\Omega \in \mathbf{B}$.

Summarising all this in a picture, we have

$$\begin{tikzcd} \mathbf{E} \arrow[r, "\mathcal{I}"] \arrow[dr, "p"'] & \mathbf{D} \arrow[r, "\mathcal{Q}"] \arrow[d, "q"] & \mathbf{B}^{\to} \arrow[dl, "cod"] \\ & \mathbf{B} & \end{tikzcd}$$

As a simple example of such a category, one can take $\mathbf{B}$ to be a topos, $\mathbf{D} = \mathbf{B}^{\to}$ and $\mathcal{Q} = id_{\mathbf{B}^{\to}}$. For $\mathbf{E}$, we take the full subcategory $Sub(\mathbf{B}) \hookrightarrow \mathbf{B}^{\to}$ of monic arrows.

In this figure, the objects of $\mathbf{E}$ are to be understood as types and the objects of $\mathbf{D}$ as kinds. For $s_1, s_2 \in \{\text{type, kind}\}$, one has that all "$(s_1, s_2)$-sums" are strong. Let's explain weak and strong sums type-theoretically: there is no difference in the formation and introduction rules.

$$\frac{\Gamma \vdash A : s_1 \qquad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Sigma x\colon A.B : s_2}\,(s_1, s_2) \qquad\qquad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : [M/x]B}{\Gamma \vdash \langle M, N\rangle : \Sigma x\colon A.B.}$$

The *weak* elimination rule is given by

$$\frac{\Gamma \vdash P : \Sigma x\colon A.B \qquad \Gamma \vdash C : s_2 \qquad \Gamma, x : A, y : B \vdash Q : C}{\Gamma \vdash Q \;\boldsymbol{where}\; \langle x, y\rangle := P : C.}$$

In the *strong* elimination rule, the type C may contain an extra variable $w : \Sigma x\colon A.B$.

$$\frac{\Gamma \vdash P : C \qquad \Gamma, w : \Sigma x\colon A.B \vdash C : s_2 \qquad \Gamma, x : A, y : B \vdash Q : [\langle x, y\rangle/w]C}{\Gamma \vdash Q \ \boldsymbol{where}\ \langle x, y\rangle := P : [P/w]C,}$$

both with conversions

$$\begin{array}{rcl} Q\ \boldsymbol{where}\ \langle x, y\rangle := \langle M, N\rangle & = & [M, N/x, y]Q \\ [\langle x, y\rangle/w]Q\ \boldsymbol{where}\ \langle x, y\rangle := P & = & [P/w]Q. \end{array}$$

6.3. PROPOSITION. (i) The elimination and conversion rules for strong (s, s)-sums can equivalently be described by the following rules with explicit projections.

$$\frac{\Gamma \vdash P : \Sigma x\colon A.B}{\Gamma \vdash \pi P : A \qquad \Gamma \vdash \pi' P : [\pi P/x]B}$$

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : [M/x]B}{\Gamma \vdash \pi\langle M, N\rangle = M : A \qquad \Gamma \vdash \pi'\langle M, N\rangle = N : [M/x]B}$$

$$\frac{\Gamma \vdash P : \Sigma x\colon A.B}{\Gamma \vdash \langle \pi P, \pi' P\rangle = P : \Sigma x\colon A.B}$$

(ii) weak (s_1, s_2)-sums $+$ strong (s_2, s_2)-sums $\Rightarrow$ strong (s_1, s_2)-sums.

Proof. (i) Standard.

(ii) Let's use "$\exists$" for the (s_1, s_2)-sums and "Σ" for the (strong) (s_2, s_2)-sums. Assume types $\Gamma \vdash A : s_1$ and $\Gamma, x : A \vdash B : s_2$ are given together with terms $\Gamma \vdash P : \exists x\colon A.B$ and $\Gamma, x : A, y : B \vdash Q : [\langle x, y\rangle/w]C$, where $\Gamma, w : \exists x\colon A.B \vdash C : s_2$. Write $C' \equiv \Sigma w\colon \exists x\colon A.B.\ C$ and $Q' \equiv \langle\langle x, y\rangle, Q\rangle$. Then $\Gamma \vdash C' : s_2$ and $\Gamma, x : A, y : B \vdash Q' : C'$. Hence one can take as new term $Q\ \boldsymbol{with}\ \langle x, y\rangle := P \equiv \pi'\{Q'\ \boldsymbol{where}\ \langle x, y\rangle := P\}$, which is of type $[P/w]C$, since

$$\begin{array}{rcl} \pi\{Q'\ \boldsymbol{where}\ \langle x, y\rangle := P\} & = & \pi\{Q'\ \boldsymbol{where}\ \langle x, y\rangle := \langle x', y'\rangle\}\ \boldsymbol{where}\ \langle x', y'\rangle := P \\ & = & \pi\{\langle\langle x', y'\rangle,\ [x', y'/x, y]Q\rangle\}\ \boldsymbol{where}\ \langle x', y'\rangle := P \\ & = & \langle x', y'\rangle\ \boldsymbol{where}\ \langle x', y'\rangle := P \\ & = & P. \ _\Box \end{array}$$

Based on this result (and on the fact that we use type-kind inclusion via the above functor $\mathcal{I}$), we conclude that a variant of *CC*-categories with weak (s, type)-sums can only have at the same time weak (type, type) and weak (kind, type) sums. Hence one can define a *weak CC*-category similarly to definition 6.2, except that $\mathcal{P}$ should only have *weak* $\mathcal{Q}$-sums — instead of the strong ones used in 6.2 (ii).

The next definition introduces a new notion. The subsequent term model example may help to convey the underlying ideas. The fact that in the calculus *HML* one can separate kind- and type-contexts plays an important structural rôle.

6.4. DEFINITION. A *HML-category* is given by the following data.

(i) A CCompC $\mathcal{Q}\colon \mathbf{D} \to \mathbf{A}^{\to}$.

(ii) A CCompC $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ over a fibration $r : \mathbf{B} \to \mathbf{A}$ (provided with terminal object functor $1 : \mathbf{A} \to \mathbf{B}$); moreover, we require that $\mathcal{P}$ admits $r^*(\mathcal{Q})$-products and strong sums.

(iii) An object $\Omega \in \mathbf{D}$ such that $q\Omega \in \mathbf{A}$ is terminal; further, the fibration p' obtained by change-of-base as below should have a generic object above $\mathcal{Q}_0\Omega \in \mathbf{A}$.

$$\begin{tikzcd}
\mathbf{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{A} \arrow[r, "1"] & \mathbf{B}
\end{tikzcd}$$

Summarising the constituents of a *HML*-category in a figure, we obtain

$$\begin{tikzcd}
\mathbf{E} \arrow[r, shift left=1.5ex] \arrow[r, shift right=1.5ex] \arrow[r, phantom, "\Downarrow \mathcal{P}"] & \mathbf{B} \arrow[d, shift right, "r"'] \\
\mathbf{D} \arrow[r, shift left=1.5ex] \arrow[r, shift right=1.5ex] \arrow[r, phantom, "\Downarrow \mathcal{Q}"] & \mathbf{A} \arrow[u, shift right, "1"']
\end{tikzcd}$$

The upper comprehension category $\mathcal{P}$ concerns the run-time part of the system; the one below concerns the compile-time part.

6.5. TERM MODEL EXAMPLE. The generic object in the definition of a *HML*-category is not as in the original system in Moggi [1991]. To obtain it in the term model we must add $\Delta \vdash \tau : \Omega \;\Leftrightarrow\; \Delta; \emptyset \vdash \tau$. We define categories as in the above figure.

A **obj.** $[\Delta]$, where Δ is a "constructor context" $\langle v_1 : k_1, \ldots, v_m : k_m \rangle$; the brackets $[-]$ denote that we take the equivalence class (wrt. conversion).

mor. $([u_1], \ldots, [u_n]) : [\Delta] \to [\langle v_1 : k_1, \ldots, v_n : k_n \rangle]$ consist of equivalence classes of constructors $\Delta \vdash u_i : [u_1, \ldots, u_{i-1}/v_1, \ldots, v_{i-1}]k_i$.

D **obj.** $[\Delta \vdash k]$.

mor. $([\vec{u}], [u']) : [\Delta \vdash k] \to [\Delta' \vdash k'] \;\Leftrightarrow\; [\vec{u}] : [\Delta] \to [\Delta']$ in $\mathbf{A}$ and $\Delta, w : k \vdash u' : [\vec{u}/\vec{v}]k'$.

B **obj.** $[\Delta; \Gamma]$, where Γ is a "term context" $\langle x_1 : \tau_1, \ldots, x_n : \tau_n \rangle$ with $\Delta;\ \langle x_1 : \tau_1,\ \ldots,\ x_{i-1} : \tau_{i-1} \rangle \vdash \tau_i : \Omega$.

mor. $([\vec{u}]; [\vec{e}]) : [\Delta; \Gamma] \to [\Delta'; \langle x_1 : \tau_1, \ldots, x_m : \tau_m \rangle] \;\Leftrightarrow\; [\vec{u}] : [\Delta] \to [\Delta']$ in $\mathbf{A}$ and $\Delta; \Gamma \vdash e_j : [e_1, \ldots, e_{j-1}/x_1, \ldots, x_{j-1}][\vec{u}/\vec{v}]\tau_j$.

E **obj.** $[\Delta; \Gamma \vdash \sigma]$.

mor. $([\vec{u}]; [\vec{e}], [e']) : [\Delta; \Gamma \vdash \sigma] \to [\Delta'; \Gamma' \vdash \sigma'] \;\Leftrightarrow\; ([\vec{u}]; [\vec{e}]) : [\Delta; \Gamma] \to [\Delta'; \Gamma']$ in $\mathbf{B}$ and $\Delta; \Gamma, y : \sigma \vdash e' : [\vec{u}/\vec{v}, \vec{e}/\vec{x}]\sigma'$.

In this way the setting is built. The presence of the required additional structure is easily verified.

7. Relating models

The relations that we are about to establish between the categorical versions of the calculi $F\omega, CC$ and HML described in the previous section, are of the following kind: given a model of calculus 1, we can perform certain categorical constructions and obtain a model of calculus 2. There will be no functoriality involved, since we did not describe appropriate morphisms between such categories.

7.1. Theorem. (i) Every *PL*-category can be transformed into a *HML*-category.
(ii) Every *HML*-category can be transformed into a *PL*-category.
(iii) The output of first applying (i) and then (ii) yields a result which is isomorphic to the input.

Proof. (i) Let $p : \mathbf{E} \to \mathbf{B}$ be a *PL*-category, i.e. a fibred CCC on a CCC $\mathbf{B}$, with a generic object and $Cons_{\mathbf{B}}$-products and sums. One forms

$$\begin{tikzcd}
\overline{\mathbf{E}} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \overline{\mathcal{P}}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\
\overline{\mathbf{B}} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow Cons_{\mathbf{B}}"] & \mathbf{B} \arrow[u, shift right, "1"']
\end{tikzcd}$$

This structure forms an *HML*-category since

- $Cons_{\mathbf{B}}$ is a CCompC, see example 4.6 (ii).
- $\overline{\mathcal{P}} : \overline{\mathbf{E}} \to \mathbf{E}^{\to}$ is a CCompC over p, see example 5.7 (ii); moreover, it has $p^*(Cons_{\mathbf{B}})$-products and strong sums by lemma 5.9.
- The generic object for p also works here, by the change-of-base situation described in 3.6.

(ii) Suppose an *HML*-category as in the figure after definition 6.4 is given. We form the fibration p'' by change-of-base

$$\begin{tikzcd}
\mathbf{E}'' \arrow[rr] \arrow[d, "p''"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbf{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E} \arrow[d, "p"] \\
\mathbf{D}_t \arrow[r, hook] & \mathbf{D} \arrow[r, "\mathcal{Q}_0"] & \mathbf{A} \arrow[r, "1"] & \mathbf{B}
\end{tikzcd}$$

where $t \in \mathbf{A}$ is terminal object. Then

- $\mathbf{D}_t$ is CCC, since $q = cod \circ \mathcal{Q}$ is a fibred CCC, see 4.11 (i).
- p'' is a fibred CCC, since fibred CCC's are preserved by change-of-base.
- The generic object T for p' above $\mathcal{Q}_0\Omega \in \mathbf{A}$ where $\Omega \in \mathbf{D}_t$ yields a generic object for p'': for every $E \in \mathbf{E}$ and $D \in \mathbf{D}_t$ with $pE = 1\mathcal{Q}_0 D$, there is a morphism $u : \mathcal{Q}_0 D \to \mathcal{Q}_0\Omega$ in $\mathbf{A}$ with $u^*(T) \cong E$ in $\mathbf{E}'$. Since $\mathcal{Q}$ is a *full* comprehension category there is a (unique) $f : D \to \Omega$ in $\mathbf{D}_t$ with $\mathcal{Q}_0 f = u$. But then we are done.

- p'' admits $Cons_{\mathbf{D}_t}$-products and sums: the essential point to verify is that p' admits $\mathcal{Q}$-products and sums; then one easily obtains that p'' admits $\widetilde{\mathcal{Q}}|_t$-products and sums — where $\widetilde{\mathcal{Q}}\,|_t$ denotes the restriction of $\widetilde{\mathcal{Q}}$ from definition 5.8 to the fibre above the terminal object t, see also remark 5.4 (i). As a special case we obtain that p'' admits $Cons_{\mathbf{D}_t}$-products and sums, since the projection $D \times D' \to D$ in $\mathbf{D}_t$ is $\widetilde{\mathcal{Q}}|_t(D, \mathcal{Q}D^*(D')) : \Sigma_D.\mathcal{Q}D^*(D') \to D$.

(iii) By the change-of-base situation $p \to \overline{p}$ from 3.6 and the fact that $\overline{\mathbf{B}}_t \cong \mathbf{B}$. 2

7.2. Theorem. (i) Every *PL*-category can be transformed into a *CC*-category.
(ii) Every *CC*-category can be transformed into a *PL*-category.

Proof. (i) Let $p : \mathbf{E} \to \mathbf{B}$ be a *PL*-category. We form

$$\begin{tikzcd}
\overline{\mathbf{E}} \arrow[r, "\mathcal{I}"] \arrow[dr, "\overline{p}"'] & \bullet \arrow[r, "\mathcal{S}"] \arrow[d] & \mathbf{E}^{\to} \arrow[dl, "cod"] \\
& \mathbf{E} &
\end{tikzcd}$$

where $\mathcal{S} = p^*(Cons_{\mathbf{B}}) \cdot \overline{\mathcal{P}}$. It is a CCompC by lemmas 3.11, 4.8, 4.9 and 5.11. The first lemma 3.11 yields the functor $\mathcal{I} : \overline{p} \to cod \circ \mathcal{S}$; $\mathcal{S}\mathcal{I} \cong \overline{\mathcal{P}}$ has strong $\mathcal{S}$-sums by 5.9, 5.7 (ii) and 4.8 (ii). Hence we are done.

(ii) Assume a *CC*-category as described in definition 6.2 is given. We obtain a *PL*-category $p'' : \mathbf{E}'' \to \mathbf{D}_t$, much in the same way as in (ii) of the previous proof:

$$\begin{tikzcd}
\mathbf{E}'' \arrow[rr] \arrow[d, "p''"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \mathbf{E} \arrow[d, "p"] \\
\mathbf{D}_t \arrow[r, hook] & \mathbf{D} \arrow[r, "\mathcal{Q}_0"] & \mathbf{B}
\end{tikzcd}$$ 2

With these two theorems one can transform *HML*-categories into *CC*-categories via *PL*-categories and vice-versa. There is however a direct "canonical" way to go from *CC* to *HML*. Whether one can do similar things the other way round is not clear.

7.3. Theorem. (i) Every *CC*-category can be transformed directly into a *HML*-category.
(ii) Doing *CC* $\longrightarrow$ *HML* $\longrightarrow$ *PL* and *CC* $\longrightarrow$ *PL* yields equivalent results.

Proof. (i) Assume we have a *CC*-category as in definition 6.2. One forms

$$\begin{tikzcd}
\widetilde{\mathbf{E}} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \widetilde{\mathcal{P}}"] & \mathbf{E} \arrow[d, shift right, "p"'] \\
\mathbf{D} \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, phantom, "\Downarrow \mathcal{Q}"] & \mathbf{B} \arrow[u, shift right, "1"']
\end{tikzcd}$$

where $\mathcal{P} = \mathcal{QI}$ is a CCompC, see 6.2 (ii). Hence $\widetilde{\mathcal{P}}$ is a CCompC over p by 5.8, admitting $p^*(\mathcal{Q})$-products and strong sums by lemma 5.10. The generic object of the CC-category also works here, because of the "pseudo" change-of-base situation $\widetilde{p} \to p$ from 5.8.

(ii) Again by the "pseudo" change-of-base situation $\widetilde{p} \to p$. ▫

References

CARTMELL, J.

[1978] *Generalised Algebraic Theories and Contextual Categories*, Ph.D. thesis, Univ. Oxford.

COQUAND, T. AND HUET G.

[1988] The Calculus of Constructions, in: *Information and Computation* (1988), vol. 73, num. 2/3.

EHRHARD, TH.

[1988] A Categorical Semantics of Constructions, in: *Logic in Computer Science* (Computer Society Press, Washington, 1988) 264–273.

HYLAND, J.M.E. and PITTS, A.M.

[1989] The Theory of Constructions: categorical semantics and topos-theoretic Models, in: GRAY, J. and SCEDROV, A., (eds.), *Categories in Computer Science and Logic* (Contemp. Math. 92, AMS, Providence, 1989).

JACOBS, B.P.F.

[1990] Comprehension Categories and the Semantics of Type Dependency, manuscript.

[1991] *Categorical Type Theory*, Ph.D. thesis, Univ. Nijmegen, forthcoming

MOGGI, E.

[1991] A Category Theoretic Account of Program Modules, *Math. Struct. in Comp. Sc.* (1991), vol. 1.

PAVLOVIĆ, D.

[1990] *Predicates and Fibrations*, Ph.D. thesis, Univ. Utrecht.

PITTS, A.M.

[1989] Categorical Semantics of Dependent Types, Notes of a talk given at SRI Menlo Park and at the Logic Colloquium in Berlin.

SEELY, R.A.G.

[1984] Locally cartesian closed Categories and Type Theory, *Math. Proc. Camb. Phil. Soc.* **95** 33 – 48.

[1987] Categorical Semantics for higher order Polymorphic Lambda Calculus, *J. Symb. Log.* **52** 969 – 989.

STREICHER, TH.

[1989] *Correctness and Completeness of a Categorical Semantics of the Calculus of Constructions*, Ph.D. thesis, Univ. Passau.

TAYLOR, P.

[1987] *Recursive Domains, indexed Categories and Polymorphism* Ph.D. thesis, Univ. Cambridge.