THOUGHTS ON A BUSEMANN EQUATION

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1. INTRODUCTION

1.1. In the present talk we comment on the following equation

$$\left(u_y^2-1\right)\cdot u_{xx}-2u_x\,u_y\cdot u_{xy}+\left(u_x^2-1\right)\cdot u_{yy}=0. \qquad \textbf{(B)}$$

This equation may be viewed as a caricature of one that A. Busemann designed in investigating irrotational ***conical flows*** of compressible fluids. The same equation governs non-parametric ***maximal surfaces*** in the three-dimensional Minkowski space, as well as stream functions attached to two-dimensional flows of a ***Chaplygin gas***. It is also cognate to ***Lavrentiev-Bitsadze*** equation, and shows up in certain generalizations of geometrical optics where ***complex-valued eikonals*** are involved.

{Adolf Busemann (Lübeck 1901, Boulder 1986) was an eminent aerospace engineer and applied mathematician, and a pioneer of supersonic aerodynamics. He designed the Busemann biplane, which emits no sonic shock waves, and invented the swept wing equipping most modern aircrafts.}

1.2. Observe:

►

$$\underbrace{-(u_y^2\cdot u_{xx}-2u_xu_y\cdot u_{xy}+u_x^2\cdot u_{yy})}_{=|\nabla u|^3\times(\text{curvature of the level lines of } u)}+\underbrace{(u_{xx}+u_{yy})}_{=\Delta u}=0$$

► Equation **(B)** has a mixed ***elliptic-hyperbolic*** character. Since

$$\text{the eigenvalues of } \underbrace{\begin{bmatrix} u_y^2-1 & -u_xu_y \\ -u_xu_y & u_x^2-1 \end{bmatrix}}_{\text{coefficient matrix}} \text{ are } -1 \;\;\&\;\; u_x^2+u_y^2-1,$$

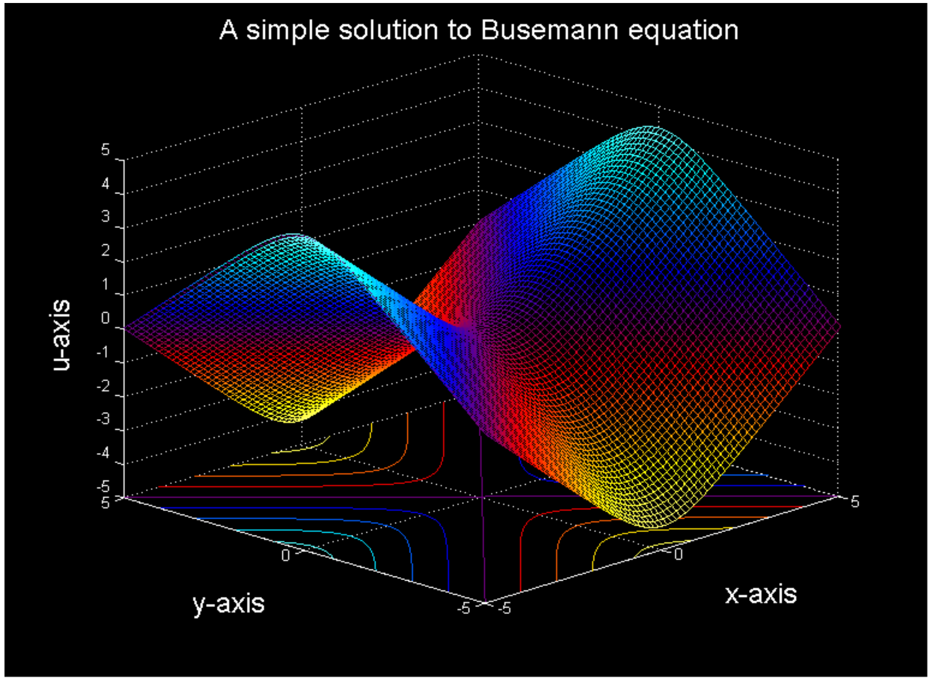
a solution u is

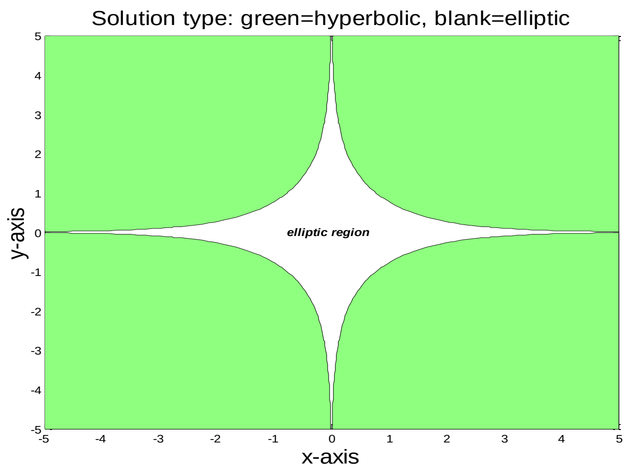
$$\text{elliptic where } u_x^2+u_y^2<1, \quad \text{hyperbolic where } u_x^2+u_y^2>1.$$

For instance, **(B)** has the following solutions

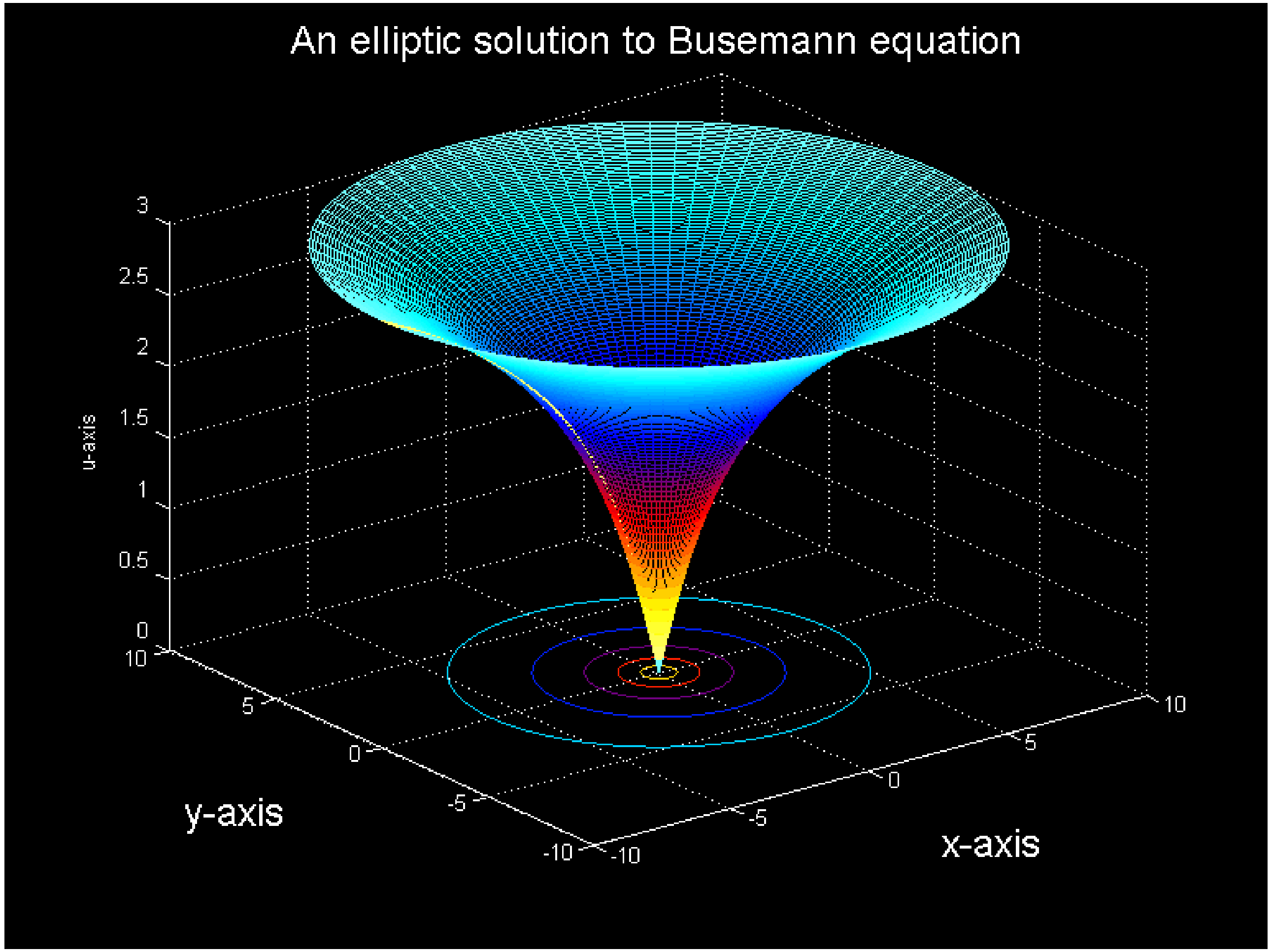
$$u(x,y)=\log\left(\frac{\cosh x}{\cosh y}\right)$$

$$(\text{elliptic where } \sinh|x|\cdot\sinh|y|<1, \quad \text{hyperbolic where } \sinh|x|\cdot\sinh|y|>1);$$

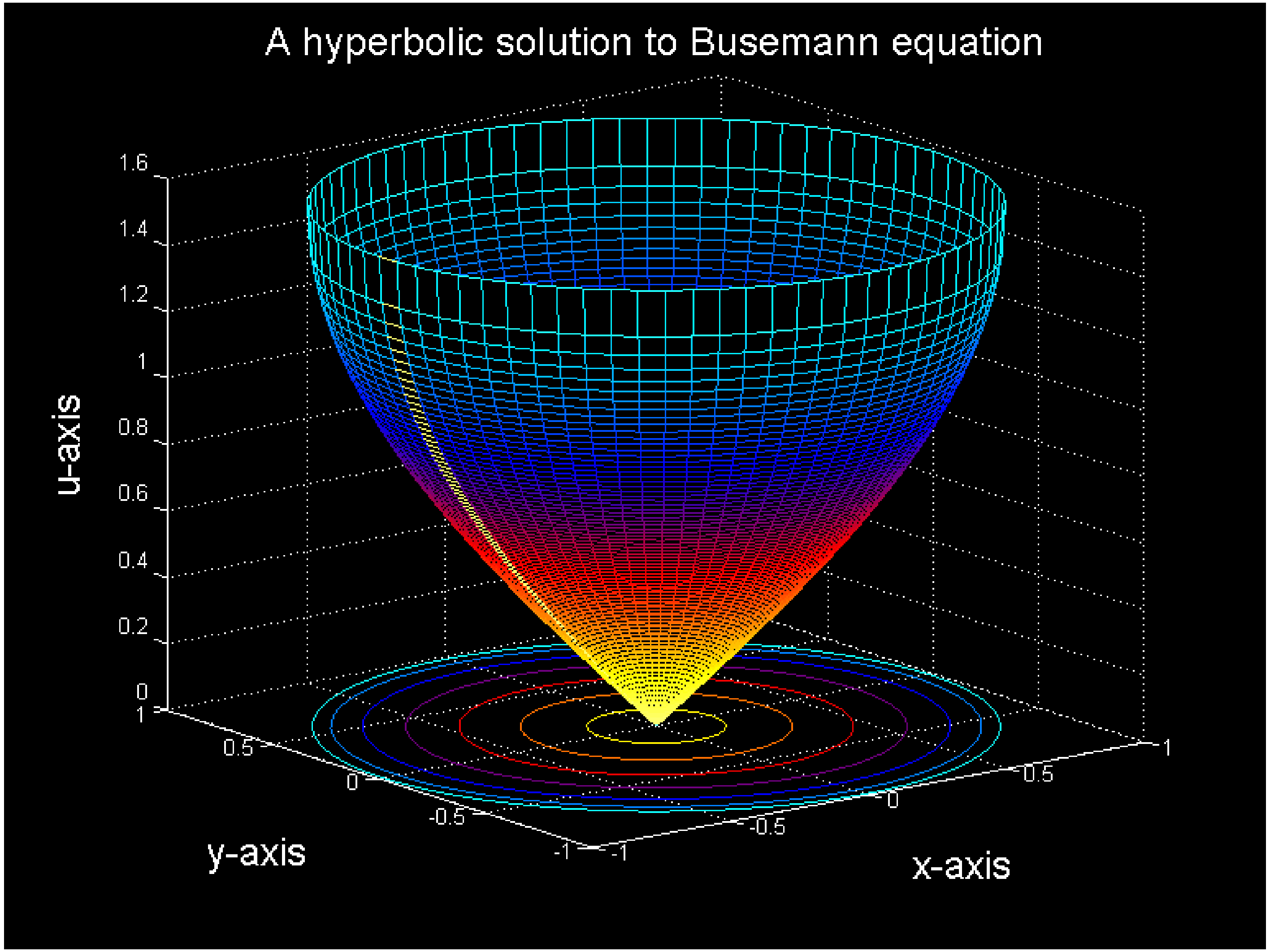

A simple solution to Busemann equation
u-axis
y-axis
x-axis


Solution type: green=hyperbolic, blank=elliptic
elliptic region
y-axis
x-axis

$$u(x, y) = \operatorname{arcsinh}\left(\sqrt{x^2 + y^2}\right) \quad \text{(elliptic everywhere);}$$



$$u(x,y) = \arcsin\left(\sqrt{x^2+y^2}\right) \quad \text{(hyperbolic everywhere).}$$

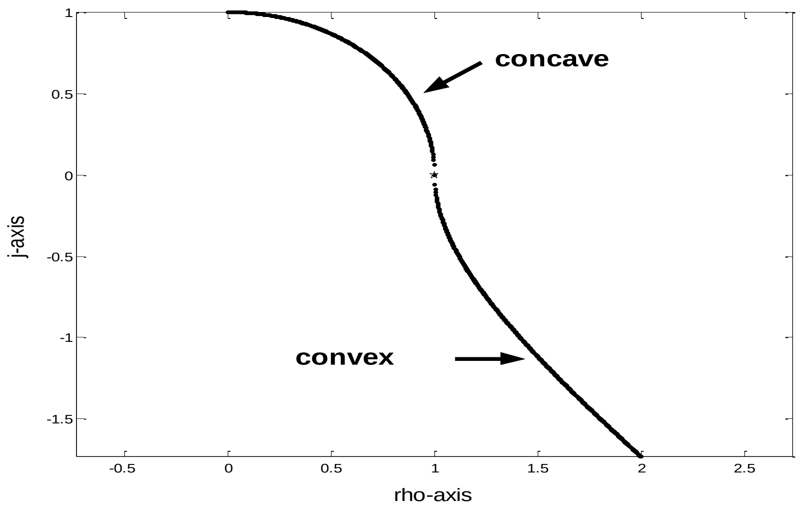


► Equation **(B)** plus the condition $u_x{}^2 + u_y{}^2 \neq 1$ can be recast in a divergence form thus

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\left|u_x^2+u_y^2-1\right|^{1/2}}\right)+\frac{\partial}{\partial y}\left(\frac{u_y}{\left|u_x^2+u_y^2-1\right|^{1/2}}\right)=0.$$

►**(B)** is formally the Euler-Lagrange equation of the variational integral defined thus

$$\underbrace{j(\rho)}_{\text{kernel}}=\left|1-\rho^2\right|^{1/2}\operatorname{sgn}(1-\rho^2)\quad (0\leq\rho<\infty),\quad J(u)=\iint \underbrace{j\left(\sqrt{u_x^2+u_y^2}\right)}_{\text{kernel}}dxdy.$$



2. MOTIVATIONS: CONICAL VELOCITY FIELDS

Luftfahrtforshung 19 (1942) 137-144

Die achsensymmetrische kegelige Überschallströmung.

Von A. Busemann, Braunschweig.

Bei Überschallgeschwindigkeit und Vernachlässigung der Reibung gibt es maßstablose Gasströmungen, d. h. solche, die sich bei beliebiger Vergrößerung oder Verkleinerung auf sich selbst abbilden. Als ebene Strömung ist die bekannte Prandtl-Meyersche »Strömung um die Ecke« von dieser Art. Zu diesen Strömungen im Raume, die auch als kegelförmige Strömungsfelder bezeichnet sind, gehört beispielsweise die gerade und die schräg angeblasene Kegelspitze. In der vorliegenden Arbeit werden die kegelförmigen Strömungsfelder mit Achsensymmetrie eingehend untersucht. Es läßt sich zeigen, daß als Parallelströmung beginnend überhaupt nur zwei Typen und deren Vereinigung möglich sind: 1. der bekannte axial angeblasene Kegel und 2. eine besondere Art von Verdichtungsdüsen. Sämtliche Strömungen weisen kegelförmige Verdichtungsstöße auf. Die Strömung an der Kegelspitze wird mit Rücksicht auf ihre ballistische Anwendung vollständig für sämtliche Kegelwinkel und Machsche Zahlen integriert. Geschlossene Ausdrücke ergeben sich nur für kleine Kegelwinkel. Das asymptotische Verhalten für Machsche Zahlen nahe eins wird besonders dargestellt.

Gliederung.

Einleitung.

Die axiale Überschallströmung gegen kegelförmige Spitzen habe ich 1929 in einem Vortrag behandelt, von dem nur ein kurzer Auszug[1]) in das Schrifttum eingegangen ist. Daher ist dieselbe Strömung teils abhängig, teils unabhängig von mehreren anderen Autoren[2]) behandelt. Eine Verallgemeinerung des hierbei auftretenden Strömungstypes habe ich 1935 in Rom auf der Volta-Tagung[3]) vorgetragen und als kegeliges Strömungsfeld bezeichnet. Obwohl durch die inzwischen erschienenen Arbeiten verschiedene einzelne Integrationen vorliegen, fehlte noch die größere Arbeit, für Luft alle möglichen Kegelwinkel und Machschen Zahlen der Anblasung so engmaschig zu integrieren, daß der Überblick vollständig wird. Diese Arbeit ist inzwischen von meinen Mitarbeitern W. Hantzsche und H. Wendt mit Hilfe des Rechenbüros meines Instituts durchgeführt. Sie haben dabei die Genauigkeit der zeichnerischen Lösungen durch numerische Integration nach dem Verfahren von Runge-Kutta kontrolliert und sich besonders um die genaue Lage der Endpunkte der Integrationslinien bemüht, die beim Zeichnen am leichtesten unsicher wird. Die von ihnen gewählten Machschen Zahlen entsprechen den Druckzahlen (999), 996, 992, 988 usw. 964, 956, 945, 930, 870,9 nach meiner Zählung im Handbuchartikel[4]).

Ich freue mich, diese Werte noch zum Gedächtnisheft des leider so früh verstorbenen C. Wieselsberger herausbringen zu können, der sich gerade in den letzten Jahren besonders stark mit Gasdynamik beschäftigt hat und in seinem Überschallkanal Kegelspitzen messen ließ.

Durch genaue Untersuchung aller möglichen kegeligen Strömungen mit Achsensymmetrie wird gezeigt, daß es neben den kegelförmigen Spitzen nur noch eine bestimmte Art von Verdichtungsdüsen gibt, deren Strömungen in diese Gruppe fallen. Dabei wird besonders eingehend dargelegt, warum diese beiden Strömungen bei vollkommenen Gasen nie völlig isentropisch verlaufen können. Eine Untersuchung darüber, wie weit die Strömungen an kegelförmigen Spitzen beobachtbar sind, wie Crocco[5]) dies für die Strömung an Keilschneiden gezeigt hat, fehlt dagegen noch.

conical fields

II. Die Differentialgleichung des kegelförmigen Feldes.

1. Wechselseitige Beziehungen zwischen Strömungsraum und Geschwindigkeitsbild. Im Gegensatz zu meinem Vortrag 1929, in dem ich die Beziehungen anschaulich entwickelt habe, möchte ich hier die analytische Ableitung des Volta-Berichtes ausführlicher wiederholen. Im Strömungsraume x, y, z soll der Strömungszustand durch ein stationäres Geschwindigkeitsfeld mit den Komponenten u, v, w festgelegt sein, wobei vorausgesetzt wird, daß für dieses Feld ein Geschwindigkeitspotential Φ besteht. Der Einwand, daß diese Voraussetzung im kegeligen Feld bei gewöhnlichen und bei vollkommenen Gasen nur selten erfüllt werden kann, ist hier unwesentlich, weil das später allein behandelte achsensymmetrische kegelige Feld für alle Gase einer dieser seltenen Fälle ist. Für das Potential liefert die Gasdynamik unter Verwendung der Schallgeschwindigkeit a folgende Differentialgleichung

Φ = velocity potential

a = velocity of sound

$$\Phi_{xx}\left(1-\frac{u^2}{a^2}\right)+\Phi_{yy}\left(1-\frac{v^2}{a^2}\right)+\Phi_{zz}\left(1-\frac{w^2}{a^2}\right)$$
$$-2\Phi_{xy}\frac{uv}{a^2}-2\Phi_{xz}\frac{uw}{a^2}-2\Phi_{yz}\frac{vw}{a^2}=0 \quad \ldots \quad (1)$$

Die unteren Indizes sind als vereinfachte Bezeichnungen für die partiellen Ableitungen nach den Raumkoordinaten angewandt. Nach der Definition des Geschwindigkeitspotentials ergeben die ersten partiellen Ableitungen nach den Raumkoordinaten die Geschwindigkeitskomponenten $\Phi_x = u$, $\Phi_y = v$, $\Phi_z = w$. Benutzt man die Geschwindigkeitskomponenten als die Koordinaten eines dreidimensionalen Geschwindigkeitsbildes (Hodograph), so ist durch die totalen Differentiale

$$\left.\begin{aligned} du &= \Phi_{xx}\,dx+\Phi_{xy}\,dy+\Phi_{xz}\,dz \\ dv &= \Phi_{yx}\,dx+\Phi_{yy}\,dy+\Phi_{yz}\,dz \\ dw &= \Phi_{zx}\,dx+\Phi_{zy}\,dy+\Phi_{zz}\,dz \end{aligned}\right\} \quad \ldots \quad (2)$$

jedem Volumenelement des Strömungsraumes mit den Kanten dx, dy, dz durch Vermittlung der zweiten Ableitungen des Potentials ein bestimmtes Volumenelement des Geschwindigkeitsbildes zugeordnet. Diese Zuordnung läßt sich umkehren, wenn die Determinante jenes linearen

[1]) A. Busemann, Drücke auf kegelförmige Spitzen bei Bewegung mit Überschallgeschwindigkeit. Z. angew. Math. u. Mech. Bd. 9 (1929), S. 496.

[2]) F. Bourquard, Ondes balistiques planes obliques et ondes coniques. Comptes Rendus Bd. 194 (1932), S. 846.

Th. v. Kármán and N. B. Moore, Resistence of slender bodies moving with supersonic velocities, with special reference to projectiles. Trans. Amer. Soc. Mech. Eng. (1932) S. 303.

G. I. Taylor and J. W. Maccoll, The air pressure on a cone moving at high speeds. Proc. R. Soc. A. Bd. 139 (1933), S. 278.

[3]) A. Busemann, Aerodynamischer Auftrieb bei Überschallgeschwindigkeit. Convegno di sc. fis. mat. e nat., Roma 1935, S. 345 und Lufo Bd. 12 (1935), S. 217.

[4]) A. Busemann, Gasdynamik in Wien-Harms, Handbuch der Experimentalphysik Bd. IV, 1, S. 426.

[5]) L. Crocco, Singolarità della corrente gassosa iperacustica nell' intorno di una prora a diedro. Aerotecnica 17 (1937), S. 519.

2.1. Consider the 3D steady irrotational flow of a perfect gas, and let

$$x, y, z = \text{space rectangular coordinates},$$

$$\varphi = \text{velocity potential},$$

$$\sigma = \text{sound speed}.$$

Standard principles of fluid dynamics yield the following equations. First,

$$\sigma = \text{a suitable function of } \nabla\varphi$$

— for instance

$$\sigma(\nabla\varphi) = L\sqrt{M^2 - |\nabla\varphi|^2} \quad (L, M = \text{Constants})$$

in the case where the flow is adiabatic and isentropic. Second,

$$\left(\sigma^2(\nabla\varphi) - \varphi_x^2\right)\cdot\varphi_{xx} + \left(\sigma^2(\nabla\varphi) - \varphi_y^2\right)\cdot\varphi_{yy} + \left(\sigma^2(\nabla\varphi) - \varphi_z^2\right)\cdot\varphi_{zz} +$$
$$- 2\varphi_x\varphi_y\cdot\varphi_{xy} - 2\varphi_x\varphi_z\cdot\varphi_{xz} - 2\varphi_y\varphi_z\cdot\varphi_{yz} = 0$$

— a quasi-linear partial differential equation of ***elliptic-hyperbolic*** type governing φ. Velocity potential φ is an ***elliptic*** solution where velocity is ***subsonic***, a ***hyperbolic*** solution where velocity is ***supersonic***.

According to Busemann, the flow is ***conical*** if

- either the set of its ***streamlines*** is invariant under homothetic transformations,
- or the ***isoclines*** of the velocity field are rays from the origin

— two equivalent conditions.

Recall

- Streamlines = lines of steepest descent of the velocity potential = orbits of

$$dx : \varphi_x = dy : \varphi_y = dz : \varphi_z \,.$$

- Isoclines = paths along which the velocity field keeps a constant direction. The isoclines are the orbits of

$$\left(\varphi_{xx} dx + \varphi_{xy} dy + \varphi_{xz} dz \right) : \varphi_x = \left(\varphi_{xy} dx + \varphi_{yy} dy + \varphi_{yz} dz \right) : \varphi_y = \left(\varphi_{xz} dx + \varphi_{yz} dy + \varphi_{zz} dz \right) : \varphi_z \,.$$

- The flow is conical if and only if the velocity potential obeys

$$\left(x\varphi_{xx} + y\varphi_{xy} + z\varphi_{xz} \right) : \varphi_x = \left(x\varphi_{xy} + y\varphi_{yy} + z\varphi_{yz} \right) : \varphi_y = \left(x\varphi_{xz} + y\varphi_{yz} + z\varphi_{zz} \right) : \varphi_z$$

— i.e. the first-order derivatives of φ are ***homogeneous*** functions of x, y, z and all have the same degree.

Didactic digression: how level lines, streamlines and isoclines look like?

For instance, if

$$\underbrace{u(x,y)}_{\text{an elliptic coordinate}} = \operatorname{arccosh}\left(\tfrac{1}{2}\sqrt{(x+1)^2+y^2}+\tfrac{1}{2}\sqrt{(x-1)^2+y^2}\right),$$

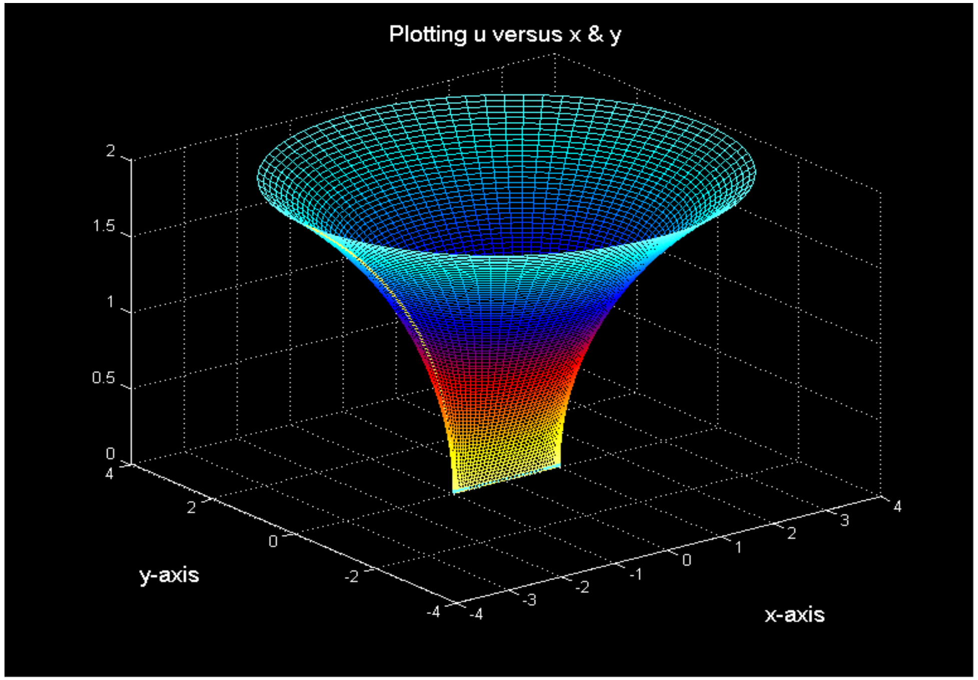
the relevant level lines, streamlines and isoclines obey

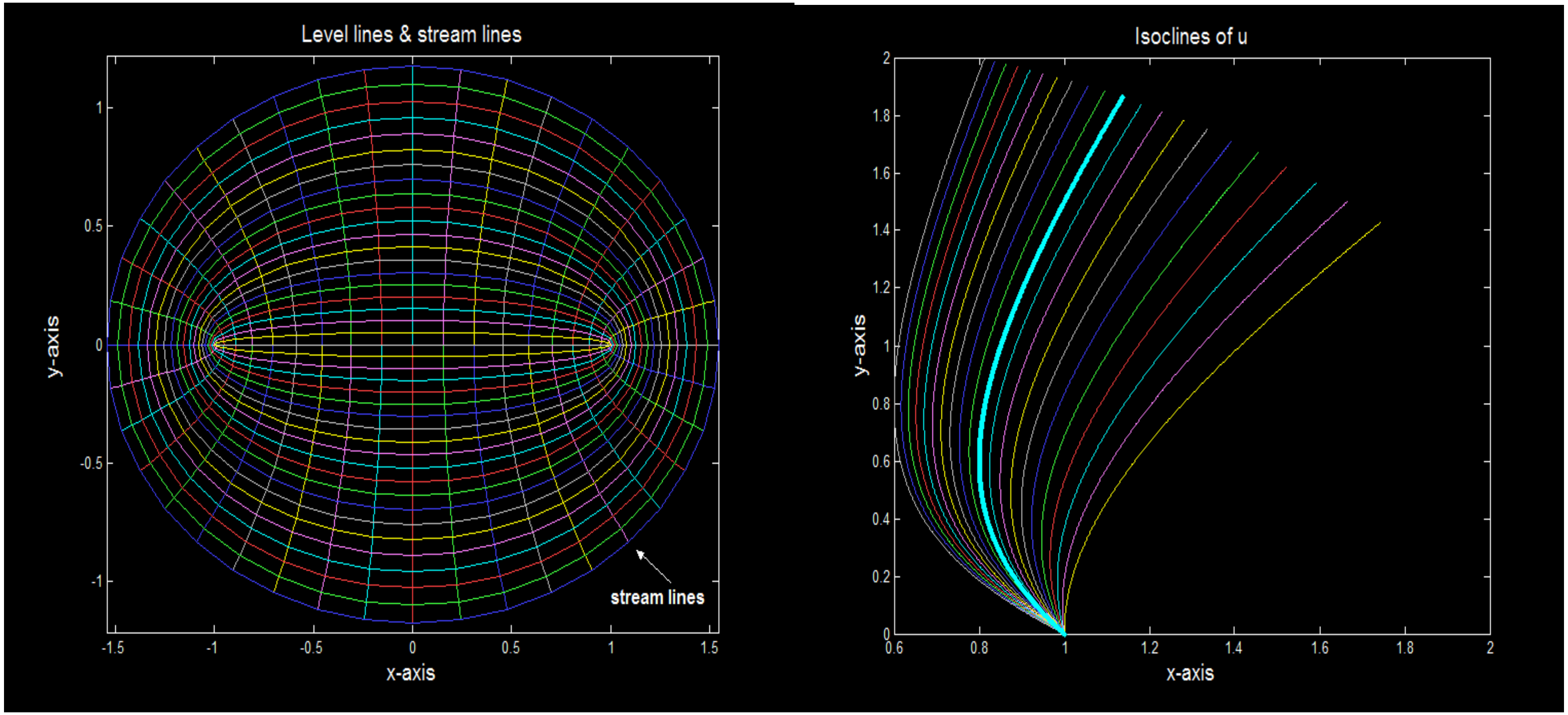
$$\frac{x^2}{\cosh^2 u}+\frac{y^2}{\sinh^2 u}=1,$$

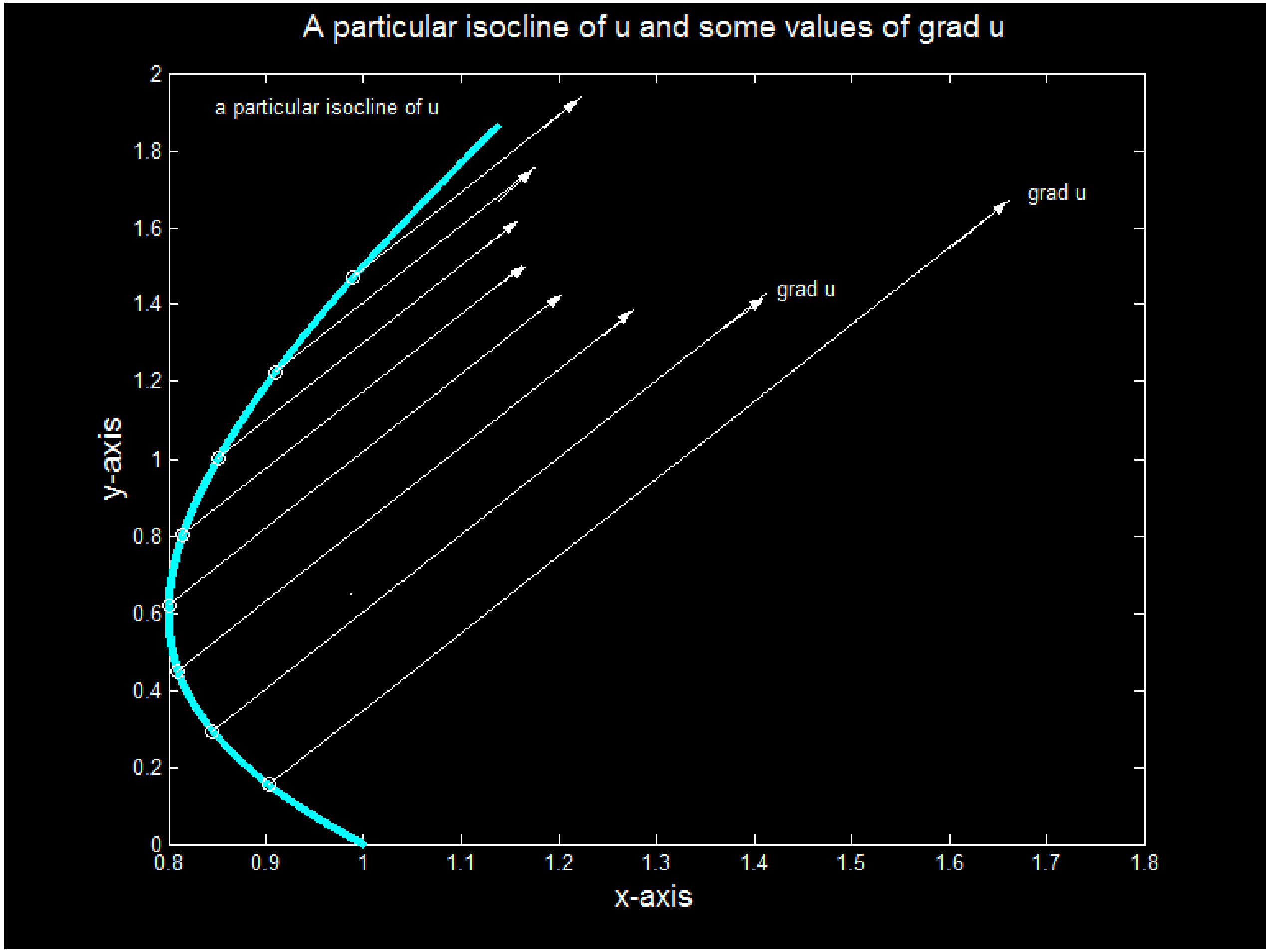
$$\frac{x^2}{\cos^2 C}-\frac{y^2}{\sin^2 C}=1 \quad (C=\text{Constant}),$$

$$x^2+2C\cdot xy-y^2=1 \quad (C=\text{Another Constant}),$$

respectively.


Plotting u versus x & y
y-axis
x-axis
2
1.5
1
0.5
0
4
2
0
-2
-4
-4
-3
-2
-1
0
1
2
3
4


Level lines & stream lines
stream lines
x-axis
y-axis
Isoclines of u
x-axis
y-axis


A particular isocline of u and some values of grad u
a particular isocline of u
grad u
grad u
y-axis
x-axis
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8

Busemann showed (in his 1942 paper displayed above) that if the flow is conical and u, v, w denote the components of the velocity, i.e. $u = \varphi_x$, $v = \varphi_y$, $w = \varphi_z$, then

- The Jacobian determinant of u, v, w vanishes identically — any component of the velocity is a function of the remaining two.
- The equation, which governs ***w as a function of u and v***, takes the form

$$\left[1-\frac{v^2}{\sigma^2}-2\frac{vw}{\sigma^2}\frac{\partial w}{\partial v}+\left(1-\frac{w^2}{\sigma^2}\right)\left(\frac{\partial w}{\partial v}\right)^2\right]\cdot\frac{\partial^2 w}{\partial u^2}+$$
$$2\left[\frac{uv}{\sigma^2}+\frac{vw}{\sigma^2}\frac{\partial w}{\partial u}+\frac{uw}{\sigma^2}\frac{\partial w}{\partial v}-\left(1-\frac{w^2}{\sigma^2}\right)\frac{\partial w}{\partial u}\frac{\partial w}{\partial v}\right]\cdot\frac{\partial^2 w}{\partial u\partial v}+$$
$$+\left[1-\frac{u^2}{\sigma^2}-2\frac{uw}{\sigma^2}\frac{\partial w}{\partial u}+\left(1-\frac{w^2}{\sigma^2}\right)\left(\frac{\partial w}{\partial u}\right)^2\right]\cdot\frac{\partial^2 w}{\partial v^2}=0\,.$$

Aerodynamicists often lower the number of independent variables by virtue of geometric or physical hypotheses. The conical-flow analysis of Busemann provides a mean of ***descending*** from a three-dimensional to a two-dimensional potential equation.

2.2. Let us call ***dimensional analysis*** into play, and ***zoom in***. If h and k obey

$$\underbrace{h}_{\text{supersonic velocity}} > \sigma(0,0,h), \qquad \underbrace{k^2}_{\text{normalnzing factor}} \cdot \left(\frac{h^2}{\sigma^2(0,0,h)} - 1 \right) = 1,$$

then

$$\text{replacing} \quad u, v, w \quad \text{by} \quad \varepsilon \cdot u, \varepsilon \cdot v, h + \varepsilon \cdot k \cdot w$$

and

$$\text{letting} \quad \varepsilon \quad \text{approach} \quad 0$$

results in

$$\left(1 - w_v^2\right) \cdot w_{uu} + 2\, w_u\, w_v \cdot w_{uv} + \left(1 - w_u^2\right) \cdot w_{vv} = 0$$

— an alias of equation **(B)**.

3. MOTIVATIONS, CONTINUED: MAXIMAL SPACE-LIKE SURFACES IN 3D MINKOWSKI SPACE

Elliptic solutions to equation **(B)** obey both

$$u_x^2 + u_y^2 < 1$$

and

$$\frac{\partial}{\partial x}\left(\frac{u_x}{(1-u_x^2-u_y^2)^{1/2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{(1-u_x^2-u_y^2)^{1/2}}\right) = 0\,,$$

therefore render an appropriate area a maximum — they can be viewed as non-parametric ***maximal space-like surfaces*** in the three-dimensional ***Minkowski space***. Recall that

$$\text{metric} = (dx)^2 + (dy)^2 - (du)^2,$$

$$\text{area of a spacelike graph} = \iint \sqrt{1-u_x^2-u_y^2}\,dxdy\,,$$

if a manifold à la Minkowski is involved.

4. Motivations, continued: Beyond geometrical optics

4.1. Let n be a real-valued, strictly positive, sufficiently smooth function of x and y — in technical words, ***n = refractive index*** of a two-dimensional, isotropic, non-conducting medium, ***1/n = velocity of propagation***. The solutions to the following equation

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \underbrace{n^2(x,y)}_{\text{refractive index}} \tag{4.1}$$

are known as two-dimensional ***eikonals***.

The theory of ***real-valued*** sufficiently smooth eikonals is the main constituent of ***geometrical optics*** (GO), which provides asymptotics for high-frequency electromagnetic fields. Two-dimensional GO ultimately amounts to manipulating:

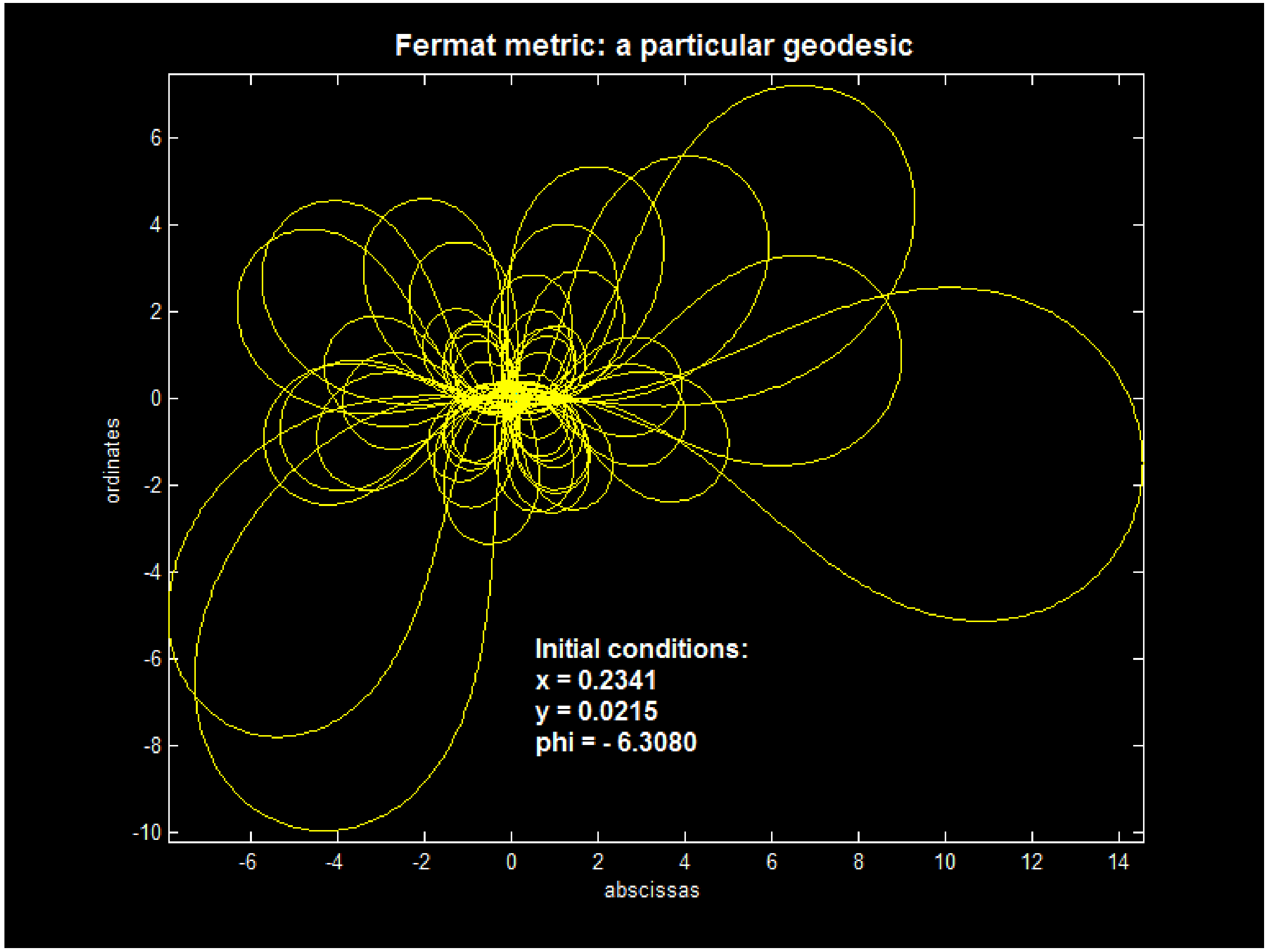
- the Riemannian metric known as ***travel time*** or ***Fermat metric***, i.e.

$$n(x,y)\sqrt{(dx)^2 + (dy)^2}\ ;$$

- appropriate one-parameter families of relevant geodesics — whose members are named ***rays***;
- the envelopes of rays — called ***caustics*****.**

Warning!

$$n(x,y) = \left(\tfrac{1}{2} + x^2 + 2y^2\right)^{-2}$$



GO-eikonals are precisely those functions whose lines of steepest descent are rays. They

- shine in light regions (those spanned by relevant rays),
- burn out beside caustics (where the ray system breaks down),
- shut down in shadow regions (the complements of light regions, which rays avoid).

On the one hand, GO is enough for successfully modeling basic optical processes — such as the propagation of light and the development of caustics. On the other hand, GO is intrinsically unable to account for optical processes that take place beyond a caustic, on the dark side of it.

Authors (Felsen, Kravtsov, Ludwig) realized that ***complex-valued*** eikonals profitably continue geometric optical eikonals into ***shadow regions***. In other words, complex-valued eikonals prove apt to account for certain optical phenomena — e.g. the rise of ***evanescent waves*** past a caustic — that are ignored by geometrical optics.

4.2. EWT is a theory of ***complex-valued*** eikonals, whose basic ingredients appear below.

Governing PDS and equations. The eikonal equation

$$\left(\partial w/\partial x\right)^2 + \left(\partial w/\partial y\right)^2 = n^2(x, y),$$

plus

$$u = \operatorname{Re} w, \quad v = \operatorname{Im} w, \quad w = u + iv,$$

result in the following system

$$\begin{aligned} &\overbrace{u_x^2 + u_y^2}^{=|\nabla u|^2} \overbrace{- v_x^2 - v_y^2}^{=-|\nabla v|^2} = n^2(x, y), \\ &\underbrace{u_x v_x + u_y v_y}_{\nabla u \cdot \nabla v} \qquad\quad = 0. \end{aligned} \tag{4.2}$$

Such a system discloses two scenarios — the former is tantamount to GO, the latter opens up a full vista of EWT. Either

$$\underbrace{u_x^2 + u_y^2 = n^2}_{\text{eikonalequation}} \quad \& \quad v_x = v_y = 0\,,$$

or the following inequalities and equations prevail

$$|\nabla u| > n \quad \& \quad |\nabla v| > 0,$$

$$\left(|\nabla u|^4 - n^2 u_y^2\right) u_{xx} + 2n^2 u_x u_y u_{xy} + \left(|\nabla u|^4 - n^2 u_x^2\right) u_{yy} - n|\nabla u|^2 \langle \nabla n | \nabla u \rangle = 0, \quad (4.3)$$

$$\left(|\nabla v|^4 + n^2 v_y^2\right) v_{xx} - 2n^2 v_x v_y v_{xy} + \left(|\nabla v|^4 + n^2 v_x^2\right) v_{yy} + n|\nabla v|^2 \langle \nabla n | \nabla v \rangle = 0, \quad (4.4)$$

$$\nabla v = f \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation}} \nabla u, \quad \underbrace{f^2 = 1 - \frac{n^2}{|\nabla u|^2}}_{\text{stretching}}, \quad \operatorname{sgn} f = \operatorname{sgn}(\underbrace{u_x v_y - u_y v_x}_{\text{Jacobian}}). \quad (4.5)$$

Peculiarities. Both (4.3) and (4.4) are quasi-linear partial differential equations of the second order.

Equation (4.3) exhibits a ***mixed elliptic-hyperbolic*** character: a solution u is

$$\text{elliptic if } |\nabla u| > n, \quad \text{hyperbolic if } |\nabla u| < n.$$

Equation (4.4) is ***elliptic-parabolic*** or ***degenerate elliptic***: a solution v such that ∇v is free from zeros is strictly elliptic, degeneracy occurs at the critical points of v.

Equations (4.5) define a ***Bäcklund transformation***, which does the following. First, it decouples solution pairs to system (4.2). Second, it maps any elliptic solution to (4.3) into an elliptic solution to (4.4), in such a way that the resulting pair satisfies system (4.2).

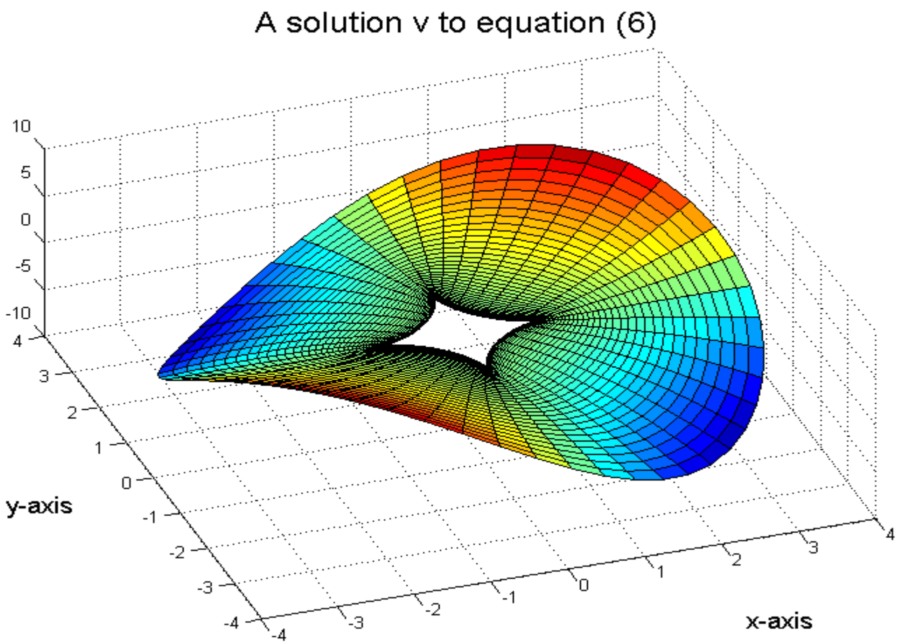
Degeneracy causes (4.4) to suffer from ***pathologies***. Relevant solutions need not possess smooth second-order derivatives, and cannot have isolated critical points. Boundary values fail to identify solutions uniquely. Solutions obeying boundary conditions happen to minimize a certain convex, coercive, variational functional — however, such a functional is not smoothly differentiable. Non-constant solutions exist that either identically equal a constant in an open region, or exhibit a continuum of extremum points.

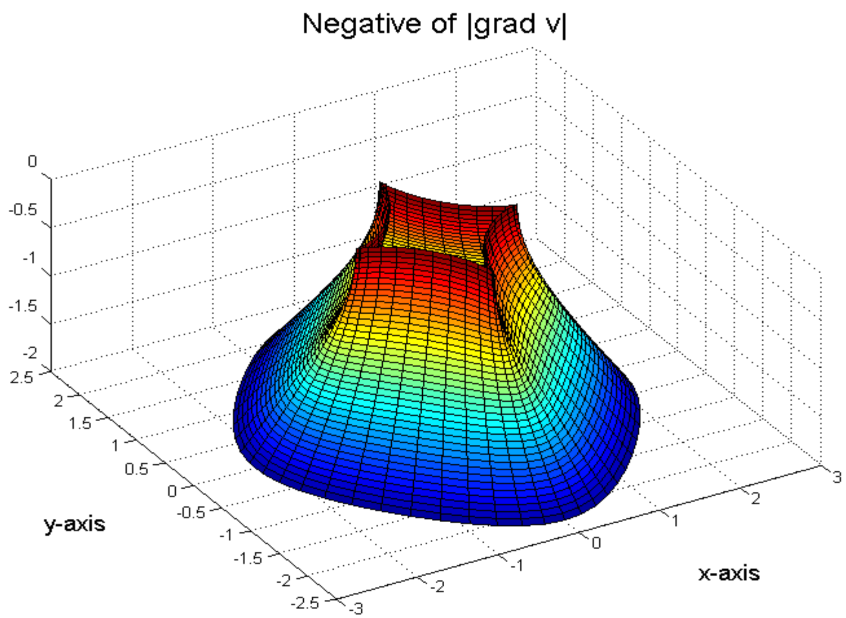
If $n \equiv 1$, the following formulas

$$x = \frac{\sin\omega}{\sqrt{1+\rho^2}}\left(\rho^2 + \sin^2\omega\right), \quad y = \frac{\cos\omega}{\sqrt{1+\rho^2}}\left(\rho^2 + \cos^2\omega\right) \quad (0 \le \rho < \infty, -\pi < \omega \le \pi),$$

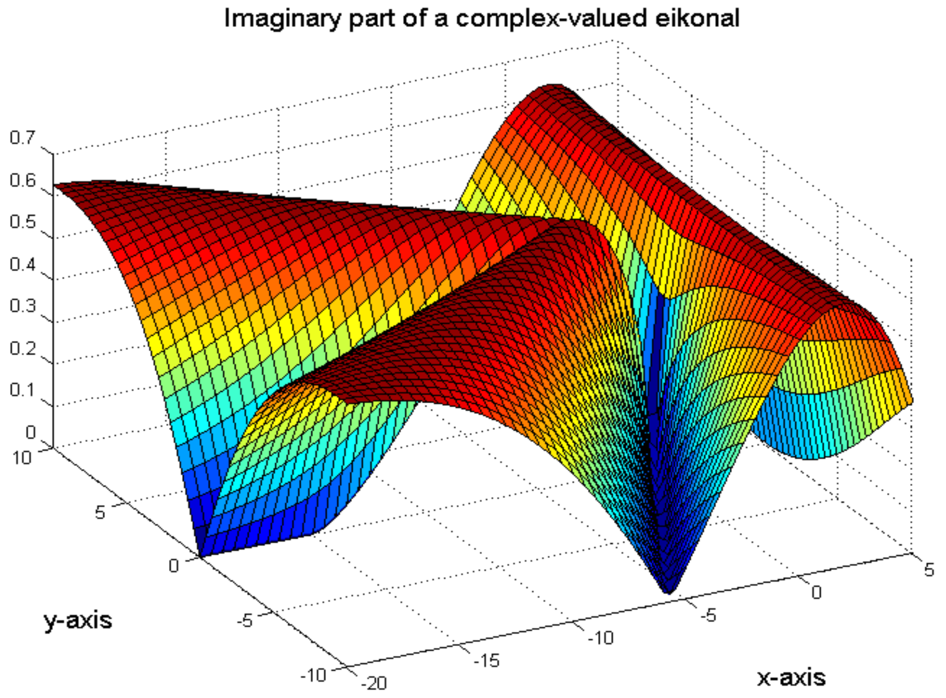
$$v = \frac{\rho^3}{2\sqrt{1+\rho^2}}\sin(2\omega), \quad v_x = \rho\cos\omega, \quad v_y = \rho\sin\omega$$

provide us with a particular solution to equation (4.4), which has ***a closed cuspidate*** ***line of critical points*** and can be ***smoothly continued by 0*** inside such a line.





A solution to equation (4.4), having ***three <u>straight-lines of critical points</u>***, is shown in the next figure.



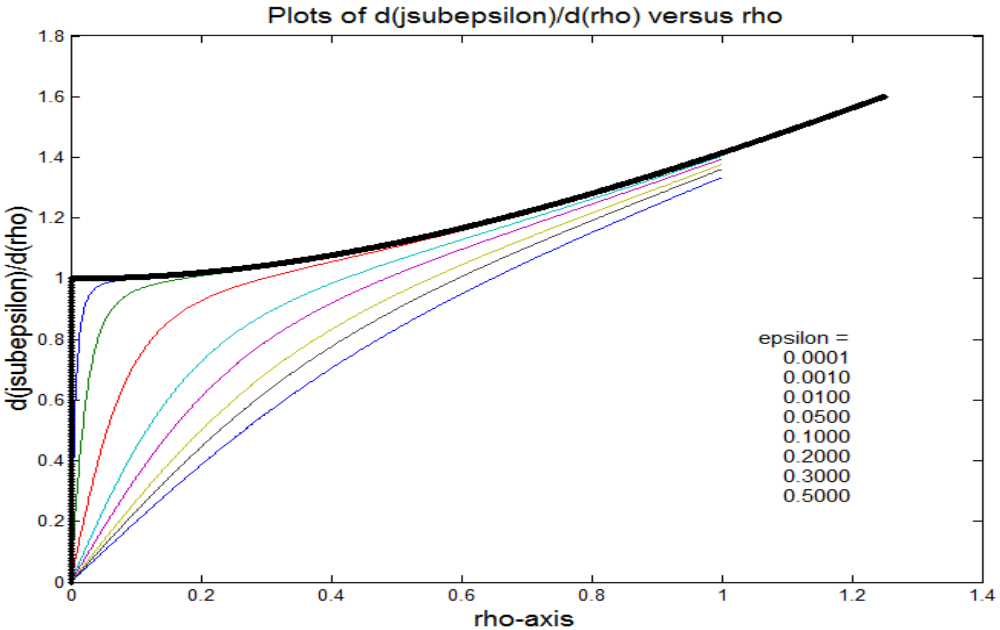
Viscosity. A ***viscosity process***, apt to overcome degeneracy, entails the following steps.

(i) Introducing a small, positive parameter ε.

(ii) Letting

$$j_\varepsilon(\rho) = \int_0^\rho t \cdot \left(\frac{1+t^2}{\varepsilon + t^2} \right)^{\frac{1}{2(1-\varepsilon)}} dt\,, \quad J_\varepsilon(v) = \iint j_\varepsilon\left(\frac{|\nabla v|}{n} \right) n^2\, dxdy\,.$$

— respectively, a kernel comme il faut and a ***smooth***, convex, and coercive functional.



(iii) Solving

$$J_\varepsilon(v) = \text{Minimum}$$

in a suitable function space and under appropriate boundary conditions — i.e. solving either

$$\operatorname{div}\left\{\left(1+\frac{1-\varepsilon}{\varepsilon+\left(\left|\nabla v\right|/n\right)^2}\right)^{\frac{1}{2(1-\varepsilon)}}\nabla v\right\}=0,$$

or

$$\varepsilon n^2\left(n^2+\left|\nabla v\right|^2\right)\Delta v+\left(\left|\nabla v\right|^4+n^2 v_y^2\right)v_{xx}-2n^2 v_x v_y v_{xy}+\left(\left|\nabla v\right|^4+n^2 v_x^2\right)v_{yy}+n\left|\nabla v\right|^2\left\langle\nabla n\middle|\nabla v\right\rangle=0$$

— two tame, ***uniformly elliptic*** partial differential equations.

(iv) Defining u via the Bäcklund transformation

$$\nabla u=\left(1+\frac{1-\varepsilon}{\varepsilon+n^{-2}\left|\nabla v\right|^2}\right)^{\frac{1}{2(1-\varepsilon)}}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\nabla v\,.$$

(v) Taking limits as ε approaches 0.

The process in hand results in a certain function pair [u v].

- The ***latter entry v*** belongs to the proper functions space, takes the prescribed boundary values and solves equation (4.4) without a hitch.

- The ***former entry u*** behaves properly where the gradient of v is essentially different from 0, however need not do the same in the complementary set. Instead of satisfying $u_x^2 + u_y^2 = n^2(x,y)$, it obeys the looser constraint

$$\left(u_y^2 - n^2\right) \cdot u_{xx} - 2u_x u_y \cdot u_{xy} + \left(u_x^2 - n^2\right) \cdot u_{yy} + \left|\nabla u\right|^2 \left\langle \nabla \log n \middle| \nabla u \right\rangle = 0 \quad (4.6)$$

in any open set where the gradient of v vanishes identically.

4.3. Equation (4.6) is the Euler-Lagrange equation of the variational integral

$$j(\rho) = 1 + \left|\rho^2 - 1\right|^{1/2} \operatorname{sgn}(\rho^2 - 1) \quad (0 \le \rho < \infty), \quad J(u) = \iint j\left(\frac{\left|\nabla u\right|}{n}\right) n^2 dxdy \,,$$

and coincides with **(B)** if $n \equiv 1$.

5. LAVRENTIEV-BITSADZE EQUATION

5.1. An embedded surface in 3D Euclidean space is qualified ***developable*** if its Gauss curvature vanishes identically. Any developable surface is either trivial (i.e. a plane, a cylinder, a cone) or is spanned by the tangent straight-lines to some saddle curve which turns into an edge of regression.

Let u be a real-valued smooth function of two real variables x and y, and let the range of ∇u be named ***hodograph*** in short. Basically, the graph of u is a developable surface if and only if the hodograph of u is either a point or a line — i.e. a function

$$[p \quad q] \mapsto f(p,q)$$

exists that satisfies

$$(\partial f/\partial p)^2 + (\partial f/\partial q)^2 > 0$$

and causes u to obey the following first-order partial differential equation

$$f(u_x, u_y) = 0.$$

Proposition. Let u be a sufficiently smooth solution to equation (B), and let the graph of u be a ***developable surface***. Then either u is linear, or u obeys

$$u_x^2 + u_y^2 = 1,$$

or else $u_x{}^2 + u_y{}^2 \geq 1$ and a constant parameter C exists such that

$$u_x \cos(C) + u_y \sin(C) = 1$$

— in other terms, the hodograph is either a ***point***, or the ***unit circle***, or else some ***tangent ray*** to the unit circle.

5.2. Suppose u is a sufficiently smooth solution to **(B)**, and the Gauss curvature of its graph is everywhere different from zero — in particular, the graph of u is definitely ***not developable***.

Then the gradient of u is locally a diffeomorphism; the ***hodograph*** of u has a non-empty interior; the first-order derivatives of u are apt to perform as ***curvilinear coordinates*** — in particular, u changes into a function of its gradient.

The ***hodograph polar coordinates*** ρ ***and*** ω, given by

$$\nabla u = \overbrace{\rho}^{\text{length}} \cdot \underbrace{\begin{bmatrix} \cos\omega \\ \sin\omega \end{bmatrix}}_{\text{slope}},$$

plus equation **(B)**, plus the hypotheses in force imply the following ***linear-looking*** equation

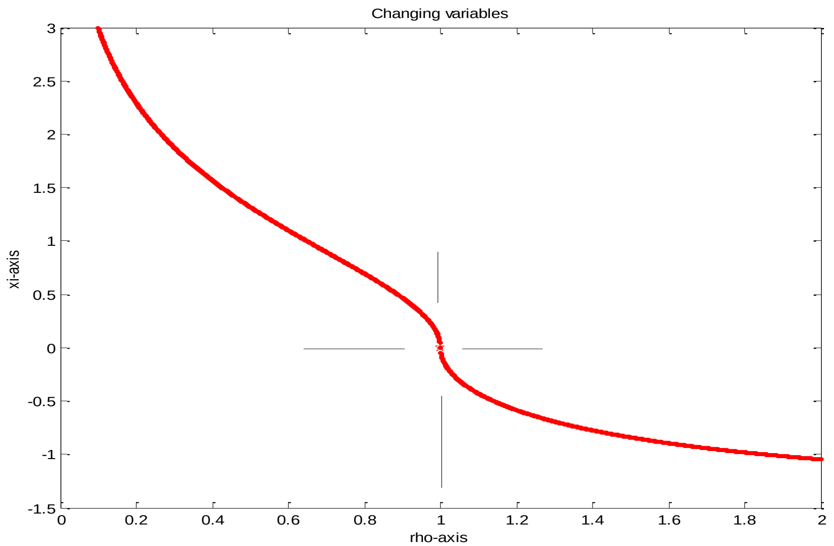
$$(1-\rho^2)\rho^2 \frac{\partial^2 u}{\partial \rho^2} + (1-2\rho^2)\rho \frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \omega^2} = 0. \qquad (5.1)$$

A new variable ξ, given by either

$$\frac{d\xi}{d\rho} = -\frac{1}{\rho\left|1-\rho^2\right|^{1/2}} \quad , \quad \xi(1) = 0,$$

or

$$\xi = \log\left(\frac{1}{\rho} + \sqrt{\frac{1}{\rho^2} - 1}\right) \quad \text{if} \quad 0 < \rho \le 1, \quad \xi = -\arccos\frac{1}{\rho} \quad \text{if} \quad 1 < \rho < \infty,$$



turns (5.1) into

$$\operatorname{sgn}(\xi)\frac{\partial^2 u}{\partial \xi^2}+\frac{\partial^2 u}{\partial \omega^2}=0 \tag{5.2}$$

— ***Lavrentiev-Bitsadze*** equation.

Lavrentiev-Bitsadze equation appears in the present context as a masked version of **(B)**.

6. LEGENDRE TRANSFORMS

6.1. Sample solutions to **(B)**, whose graph is ***not developable***, result from an appropriate use of the ***Legendre transformation***.

This transformation is instrumental in classical and statistical mechanics, thermodynamics, convex analysis, and differential equations. One of its peculiarities is turning second-order ***quasi-linear*** homogeneous partial differential equations in two independent variables, whose coefficients depend on first-order derivatives only, into surrogates that are ***linear*** and more tractable.

6.2. Let u be a real-valued, twice continuously differentiable function of two real variables x and y; suppose the gradient of u is a ***bijection***, and the graph of u has a ***non-vanishing Gauss curvature***. A recipe for handling the ***Legendre transform U of u*** reads thus. Let

$$\underbrace{(p,q)}_{\text{slope}} = \text{any pair from the range of } \nabla u$$

— ***a slope*** — and consider the tangent plane to the graph of u that is orthogonal to

$$\underbrace{\begin{bmatrix} p \\ q \\ -1 \end{bmatrix}}_{\text{normal vector}};$$

then

$$\underbrace{-U(p,q)}_{\text{intercept}} = \text{the height of such a plane above the origin}$$

— ***an intercept***. In other words, the triple

$$p,\ q, \text{ value of } U \text{ at } (p,q)$$

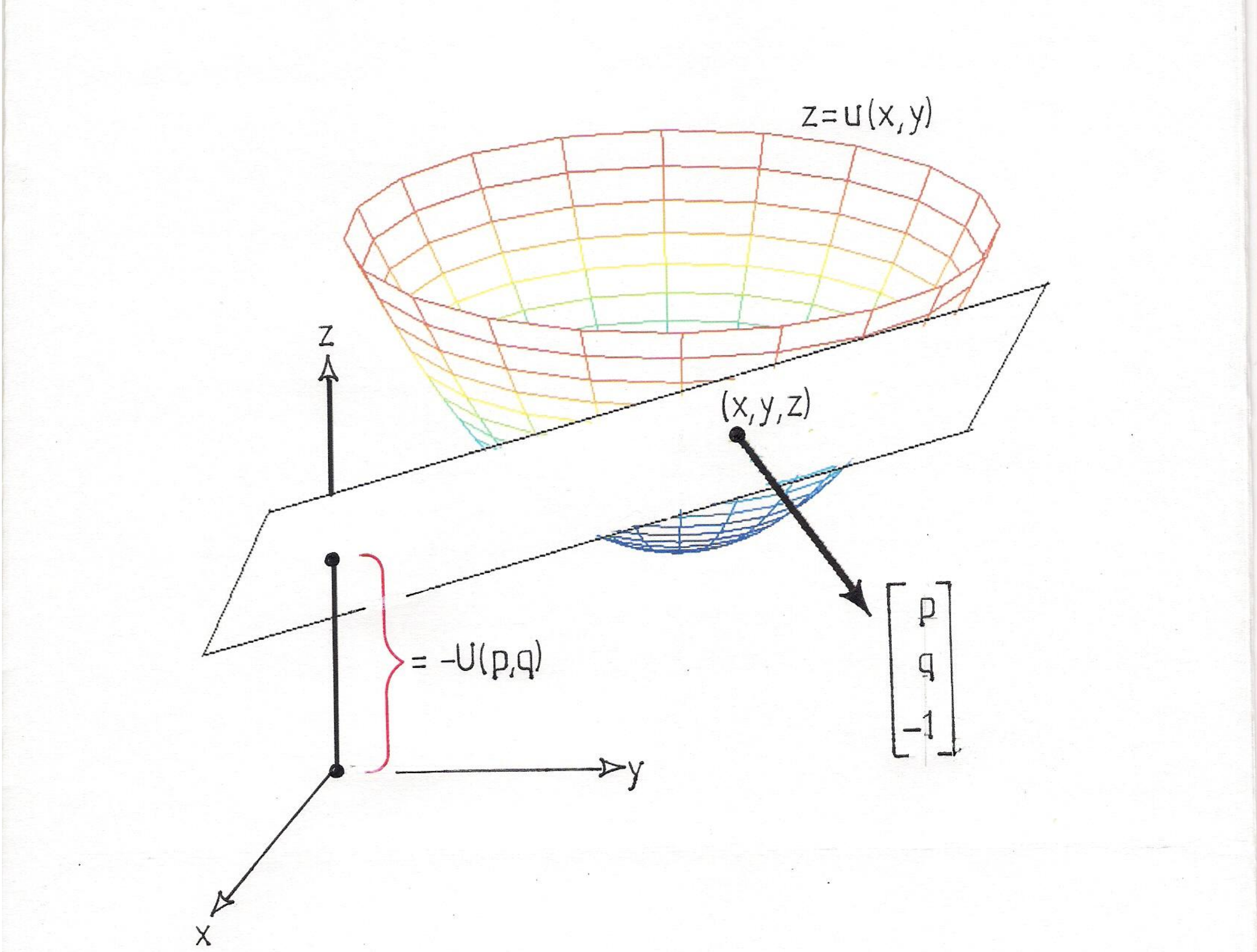
detects any tangent plane to the graph of u. The following formulas result

$$p = \frac{\partial u}{\partial x}(x,y), \quad q = \frac{\partial u}{\partial y}(x,y),$$

$$xp + yq = u(x,y) + U(p,q),$$

$$x = \frac{\partial U}{\partial p}(p,q), \quad y = \frac{\partial U}{\partial q}(p,q),$$

and provide a parametric representation of both u and U — parameters coincide with relevant first-order derivatives.



z=u(x,y)
z
(x,y,z)
= −U(p,q)
p
q
−1
y
x

6.3. If u obeys **(B)** and the above hypotheses, then the Legendre transform U of u obeys

$$(p^2-1)\frac{\partial^2 U}{\partial p^2}+2pq\frac{\partial^2 U}{\partial p\partial q}+(q^2-1)\frac{\partial^2 U}{\partial q^2}=0. \qquad (6.1)$$

- Equation (6.1) is

$$\textbf{\textit{elliptic}} \text{ where } \quad p^2+q^2<1, \quad \textbf{\textit{hyperbolic}} \text{ where } \quad p^2+q^2>1;$$

- the ***characteristic lines*** in the hyperbolic region are tangent rays to the unit disk.

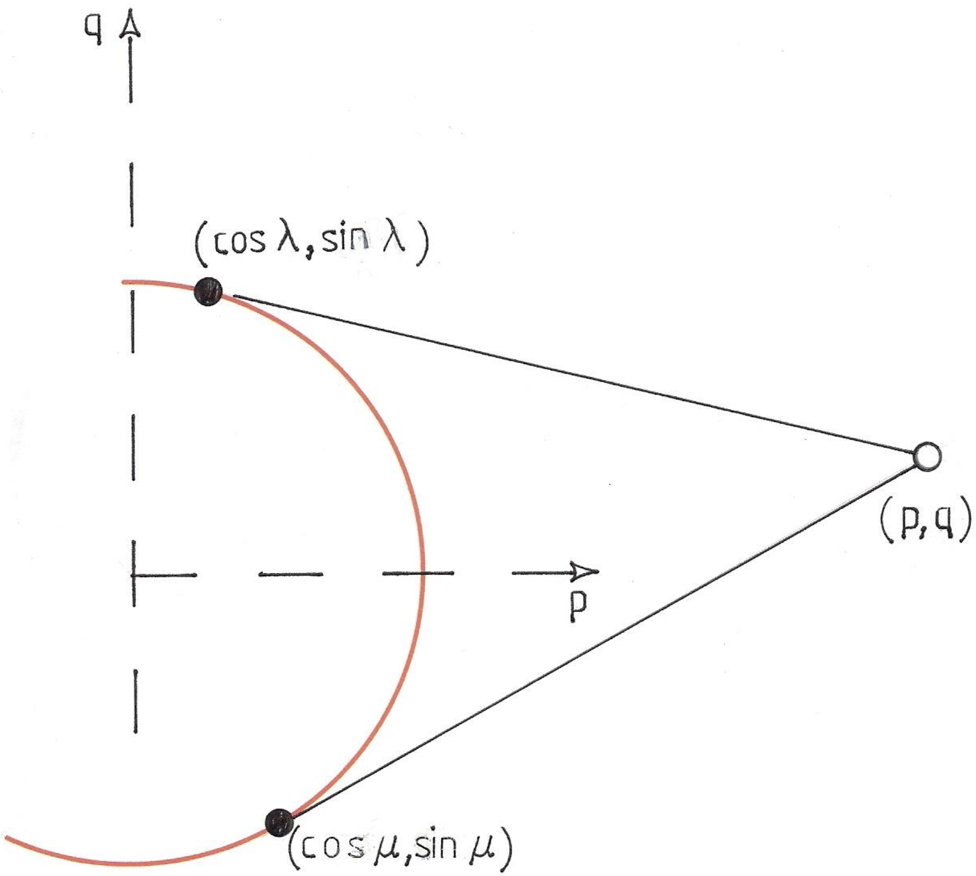
If ***characteristic coordinates*** λ ***&*** μ are introduced in the hyperbolic region according to

$$p=\frac{\cos\frac{\lambda+\mu}{2}}{\cos\frac{\lambda-\mu}{2}}, \quad q=\frac{\sin\frac{\lambda+\mu}{2}}{\cos\frac{\lambda-\mu}{2}},$$

$$p\cos\lambda+q\sin\lambda=1, \quad p\cos\mu+q\sin\mu=1,$$

$$p = \frac{\cos\dfrac{\lambda+\mu}{2}}{\cos\dfrac{\lambda-\mu}{2}}, \quad q = \frac{\sin\dfrac{\lambda+\mu}{2}}{\cos\dfrac{\lambda-\mu}{2}};$$

$$p\cos\lambda + q\sin\lambda = 1, \quad p\cos\mu + q\sin\mu = 1.$$



then in the hyperbolic region equation (6.1) takes the following form

$$\frac{\partial^2 U}{\partial\lambda\partial\mu} - \frac{1}{\sin(\lambda-\mu)}\left(\frac{\partial U}{\partial\lambda} - \frac{\partial U}{\partial\mu}\right) = 0 \qquad (6.2)$$

— somewhat reminiscent of the ***Euler-Poisson-Darboux equation***.

If ***hodograph polar coordinates*** ρ **&** ω are in use, (6.1) takes the following form

$$(1-\rho^2)\frac{\partial^2 U}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\omega^2} = 0. \qquad (6.3)$$

6.4. Sample solutions to (6.1) result from routine devices. Separating variables in equation (6.3) gives

$$\underbrace{p=\rho\cos\omega,\quad q=\rho\sin\omega,}_{\text{hodograph polar coordinates}}\quad U(p,q)=\underbrace{R(\rho)\cdot\cos(k\omega)}_{\text{separation of variables}}\quad (k=0,\pm1,\pm2,\pm3,\cdots),$$

$$R(\rho)=\underbrace{\rho^{k}Q(z),\quad z=\rho^{2},}_{\text{change of variables}}$$

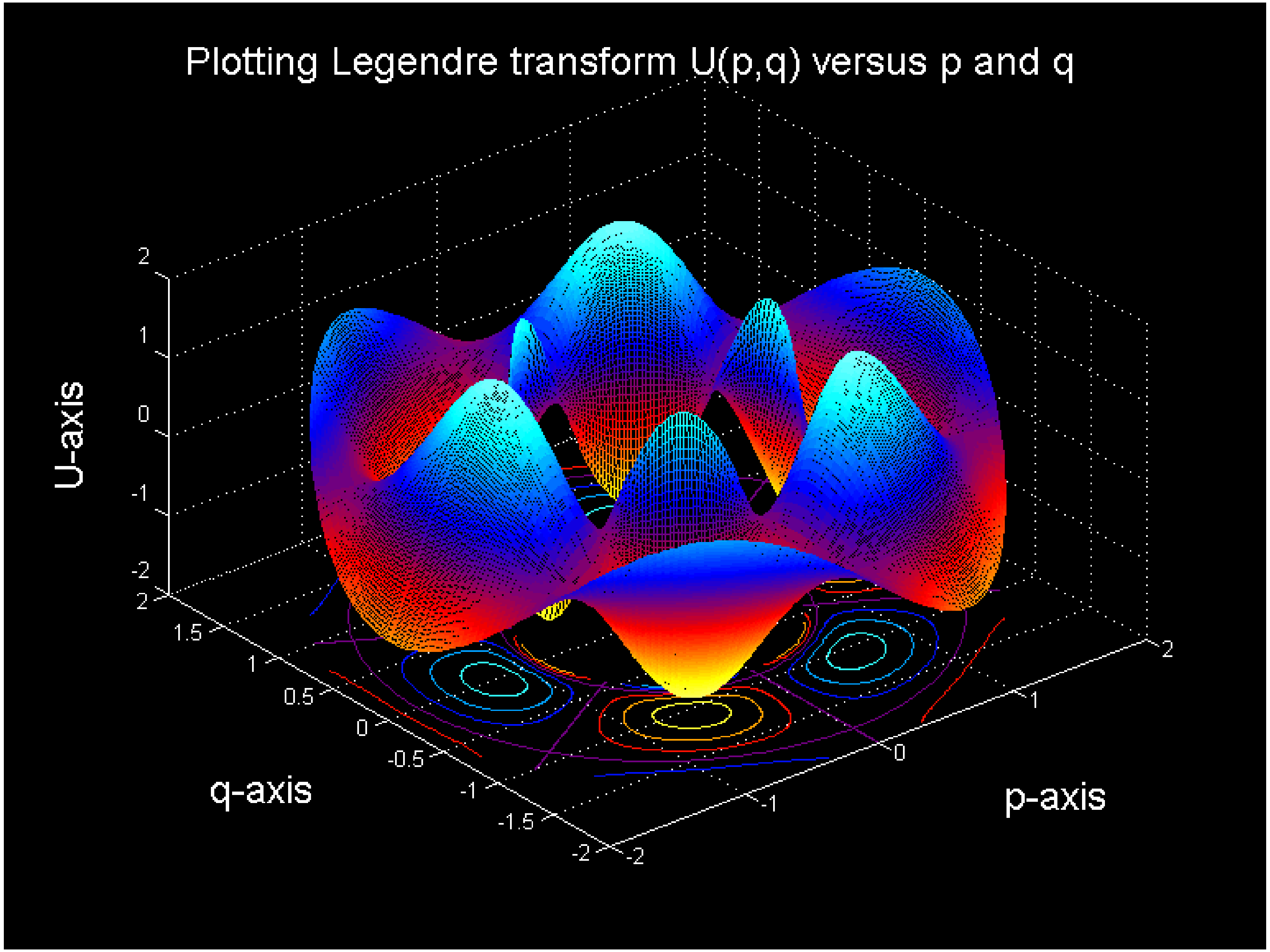
$$\underbrace{z(1-z)Q''+\left[k+1-(k+\tfrac{1}{2})z\right]Q'-\frac{k(k-1)}{4}Q=0,}_{\text{hypergeometric differential equation}}$$

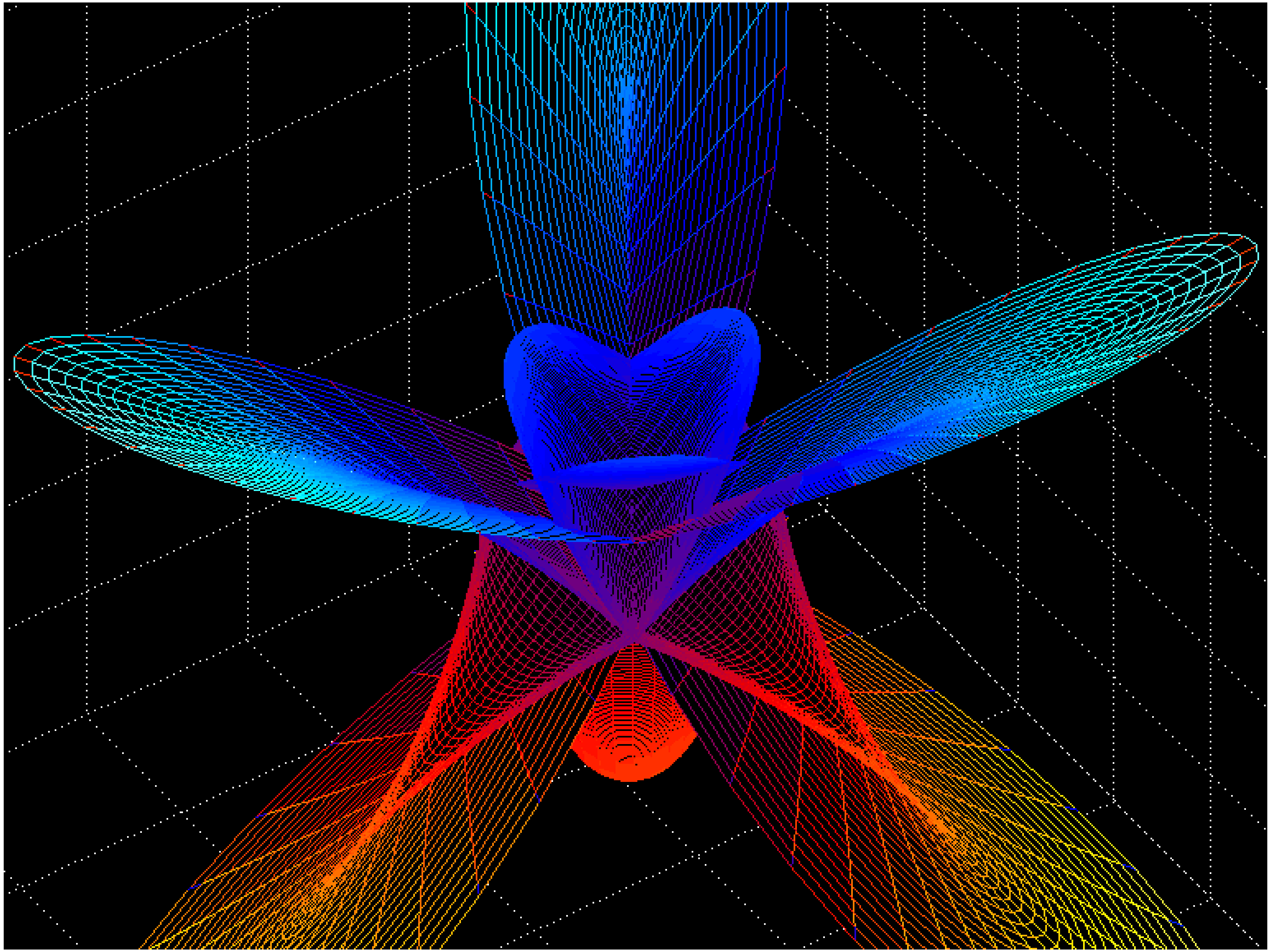
$$Q(z)=\underbrace{F\left(\frac{k}{2},\frac{k-1}{2};k+1;z\right).}_{F(a,\,b;\,c;\,z)\,=\,\text{hypergeometric function, see [AS, Chapter 15].}}$$

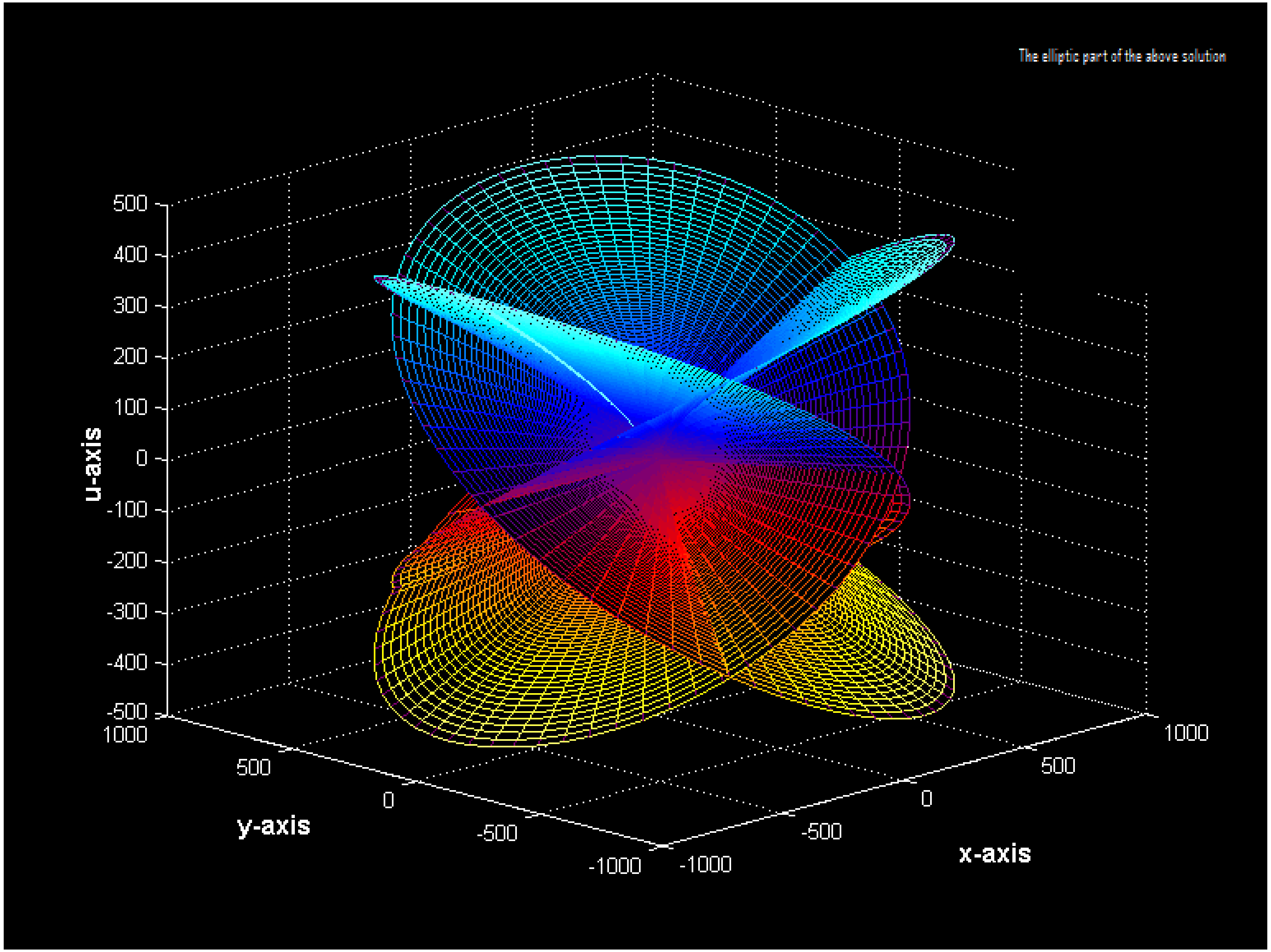
For instance, a solution to **(B)** having a mixed ***elliptic-hyperbolic*** character results from

$$p=\rho\cos\omega,\quad q=\rho\sin\omega,\quad U(p,q)=\left(3\rho-\frac{12}{\rho}+\frac{8}{\rho^{3}}\right)\cos(3\omega),$$

and is shown in the next figures.


Plotting Legendre transform U(p,q) versus p and q
U-axis
q-axis
p-axis
2
1
0
-1
-2
1.5
0.5
-0.5
-1.5




The elliptic part of the above solution
u-axis
500
400
300
200
100
0
-100
-200
-300
-400
-500
y-axis
1000
500
0
-500
-1000
x-axis
-1000
-500
0
500
1000

7. DIGRESSING ON BÄCKLUND TRANSFORMATIONS

7.1 Loosely speaking, a ***Bäcklund transformation*** converts a solution to some partial differential equation into a different solution to the same equation, or into a solution to another partial differential equation. Such a transformation allows an extra solution to a partial differential equation to come out if one particular solution to the same or another equation is in hand. A Bäcklund transformation typically looks like a first-order partial differential system, which relates two functions in a convenient way and drives them to obey partial differential equations individually. Bäcklund transformations may be of considerable service; however, no systematic way of finding them is available. These transformations trace back to works by ***L. Bianchi*** and ***A.V. Bäcklund*** in differential geometry, and play a role especially in soliton theory and integrable systems. They come up in gas dynamics too, and are a key to the present work.

7.2. Miscellaneous examples follow.

(i) If a_{11} , a_{12} , a_{21} , a_{22} are bounded and measurable functions of x and y and their matrix is non-singular, then the transformation attached to the equation

$$\nabla v = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \nabla u$$

generalizes ***Cauchy Riemann equations***. It maps any (suitably smooth) solution to

$$\operatorname{div}\left\{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\nabla u\right\}=0$$

into a solution to

$$\operatorname{div}\left\{\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T\nabla v\right\}=0$$

that obeys the orthogonality condition

$$\left\langle \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\nabla u \middle| \nabla v \right\rangle = 0.$$

For instance, the following Bäcklund transformation

$$v_x = -u_y\ , \quad v_y = y\cdot u_x\ ,$$

convert any solution of

$$y\cdot u_{xx} + u_{yy} = 0$$

— the ***Tricomi equation*** — into a solution of

$$v_{xx}+\frac{\partial}{\partial y}\frac{v_y}{y}=0\,,$$

which obeys

$$y\cdot u_x v_x + u_y v_y = 0.$$

(ii) Let a_{11} , a_{12} , a_{21} , a_{22} be bounded and measurable functions of x and y such that

$$a_{11}\neq 0 \quad \& \quad a_{22}\neq 0,$$

and let φ obey

$$a_{11}\varphi_{xx}+2a_{12}\varphi_{xy}+a_{22}\varphi_{yy}=0.$$

If

$$\begin{bmatrix} u \\ v \end{bmatrix}=\nabla\varphi\,,$$

then u and v are related by the following transformations

$$\nabla v=-\frac{1}{a_{22}}\begin{bmatrix} 0 & -a_{22} \\ a_{11} & 2a_{12} \end{bmatrix}\nabla u,\quad \nabla u=-\frac{1}{a_{11}}\begin{bmatrix} 2a_{12} & a_{22} \\ -a_{11} & 0 \end{bmatrix}\nabla v,$$

and obey

$$\operatorname{div}\left\{\frac{1}{a_{22}}\begin{bmatrix} a_{11} & 2a_{12} \\ 0 & a_{22}\end{bmatrix}\nabla u\right\}=0, \quad \operatorname{div}\left\{\frac{1}{a_{11}}\begin{bmatrix} a_{11} & 0 \\ 2a_{12} & a_{22}\end{bmatrix}\nabla v\right\}=0.$$

(iii) Suppose $0 \le \rho \mapsto j(\rho)$ is a smooth real-valued function, and $j'(0)=0$. Let u be an ***extremal*** of the ***variational integral***

$$\iint j\left(\sqrt{u_x^2+u_y^2}\right) dxdy\,,$$

i.e. a solution to

$$\frac{\partial}{\partial x}\left\{ j'\left(\sqrt{u_x^2+u_y^2}\right)\frac{u_x}{\sqrt{u_x^2+u_y^2}}\right\}+\frac{\partial}{\partial y}\left\{ j'\left(\sqrt{u_x^2+u_y^2}\right)\frac{u_y}{\sqrt{u_x^2+u_y^2}}\right\}=0\,,$$

and let ρ and ω be the ***hodograph polar coordinates*** given by

$$\nabla u=\rho\cdot\begin{bmatrix}\cos\omega \\ \sin\omega\end{bmatrix}.$$

The following equation holds

$$\nabla\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} \rho j''(\rho)/j'(\rho) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \frac{\nabla\rho}{\rho}$$

— in other words, a Bäcklund transformation relates ρ and ω.

(iv) The transformation attached to the following formula

$$u = \log\left(2\frac{v_x^2 + v_y^2}{v^2} \right)$$

maps solutions to

$$\Delta v = 0$$

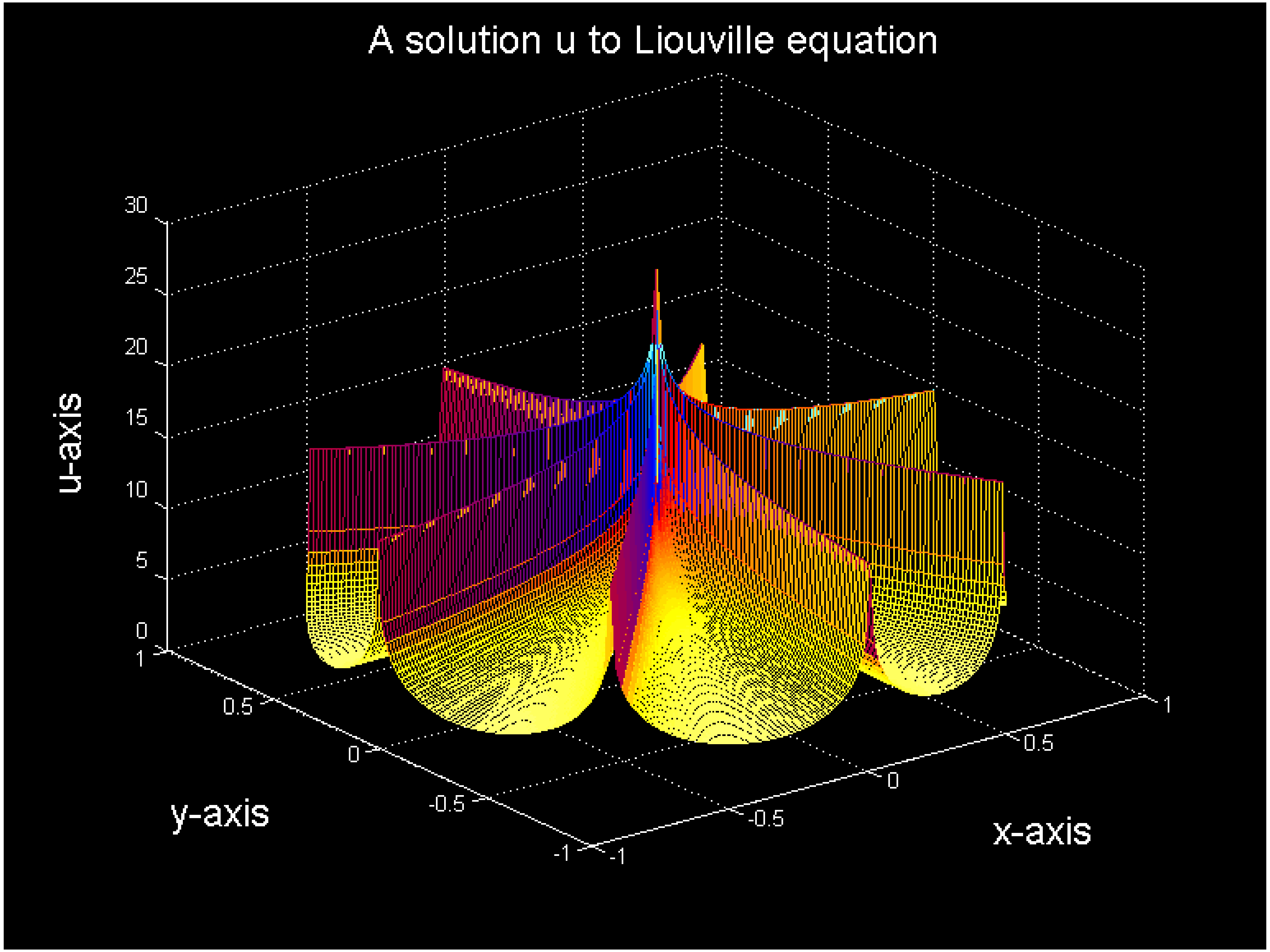
(***Laplace equation***) into solutions to

$$\Delta u = \exp(u)$$

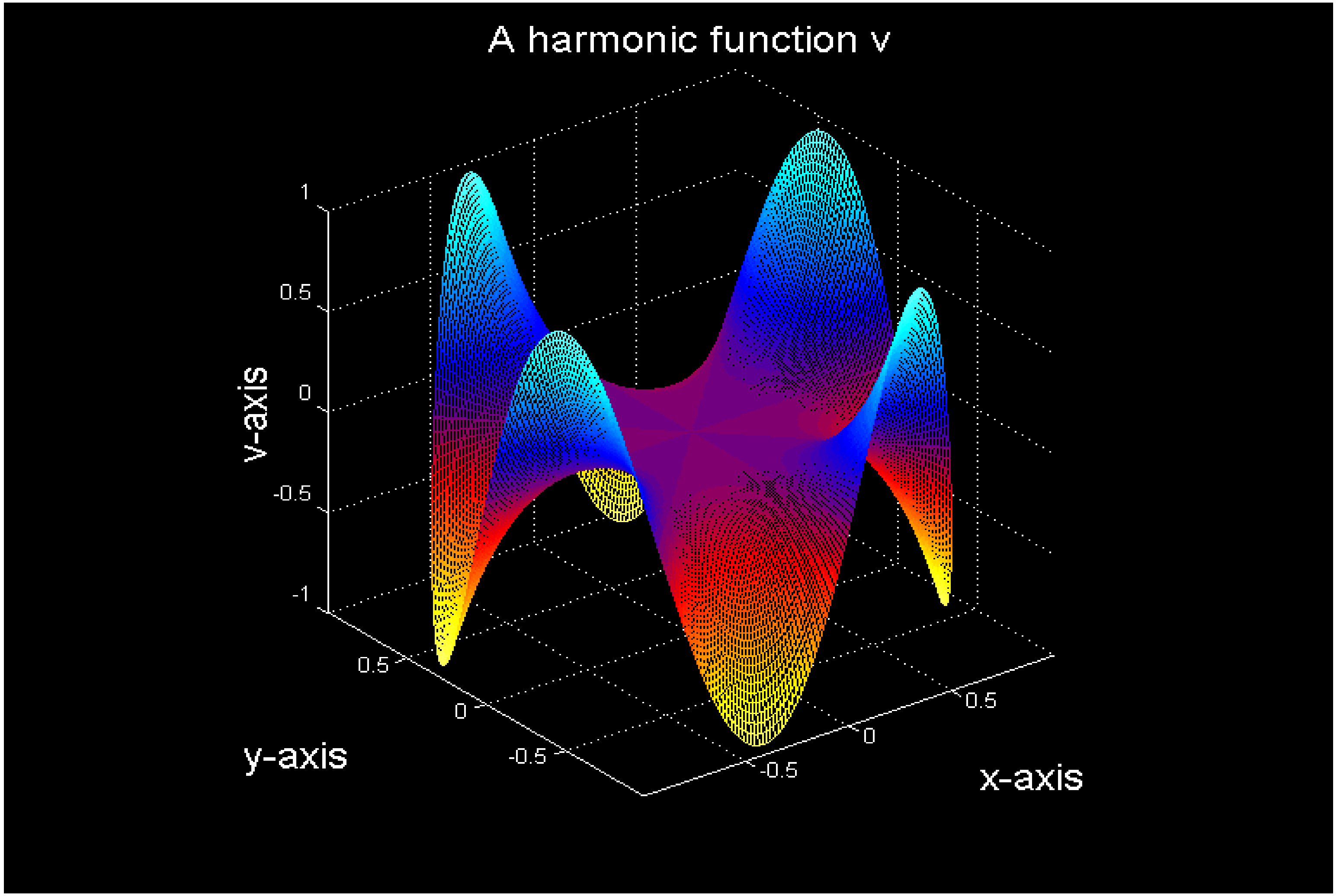
(***Liouville equation***).

Here is a sample pair

$$u(x,y) = \log\frac{32(x^2+y^2)^3}{x^2y^2(x^2-y^2)^2}, \quad v(x,y) = xy(x^2-y^2).$$



A solution u to Liouville equation
u-axis
30
25
20
15
10
5
0
1
0.5
0
-0.5
-1
y-axis
-1
-0.5
0
0.5
1
x-axis


A harmonic function v
1
0.5
0
-0.5
-1
v-axis
0.5
0
-0.5
y-axis
-0.5
0
0.5
x-axis

(v) The transformations attached to the following formulas

$$\nabla u = (1+v^2)^{-1/2}\begin{bmatrix}1\\ v\end{bmatrix}, \quad v = \frac{u_y}{u_x},$$

are inverse of one another. They convert any (suitably smooth) solution *u* to

$$u_x^2 + u_y^2 = 1$$

(***eikonal equation of geometrical optics***) into a solution *v* to

$$v_x + v v_y = 0$$

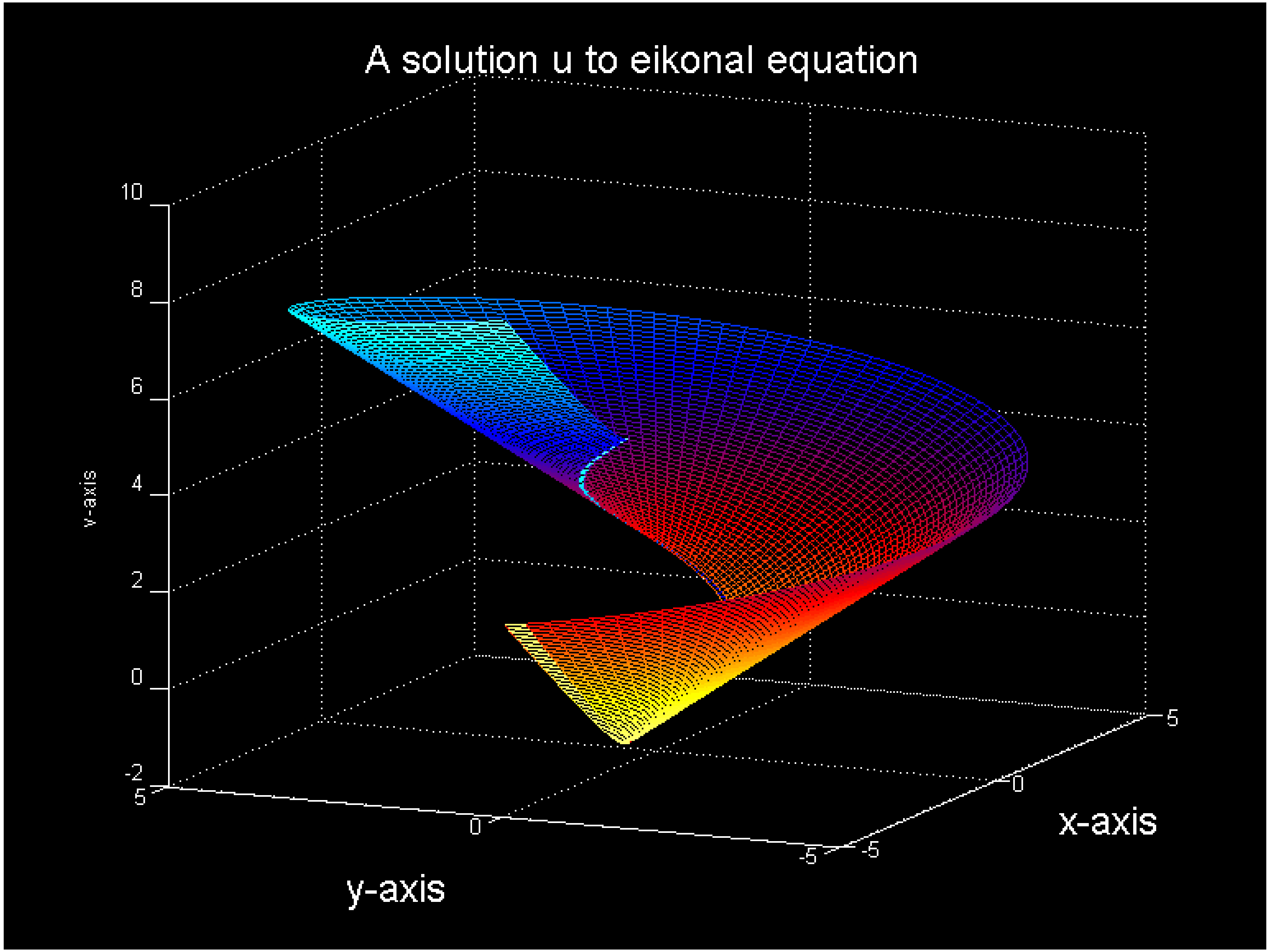
(***inviscid Burgers equation***), and vice versa — the level-lines and the shock-line of *v* are the isoclines and the caustic of *u*, respectively.

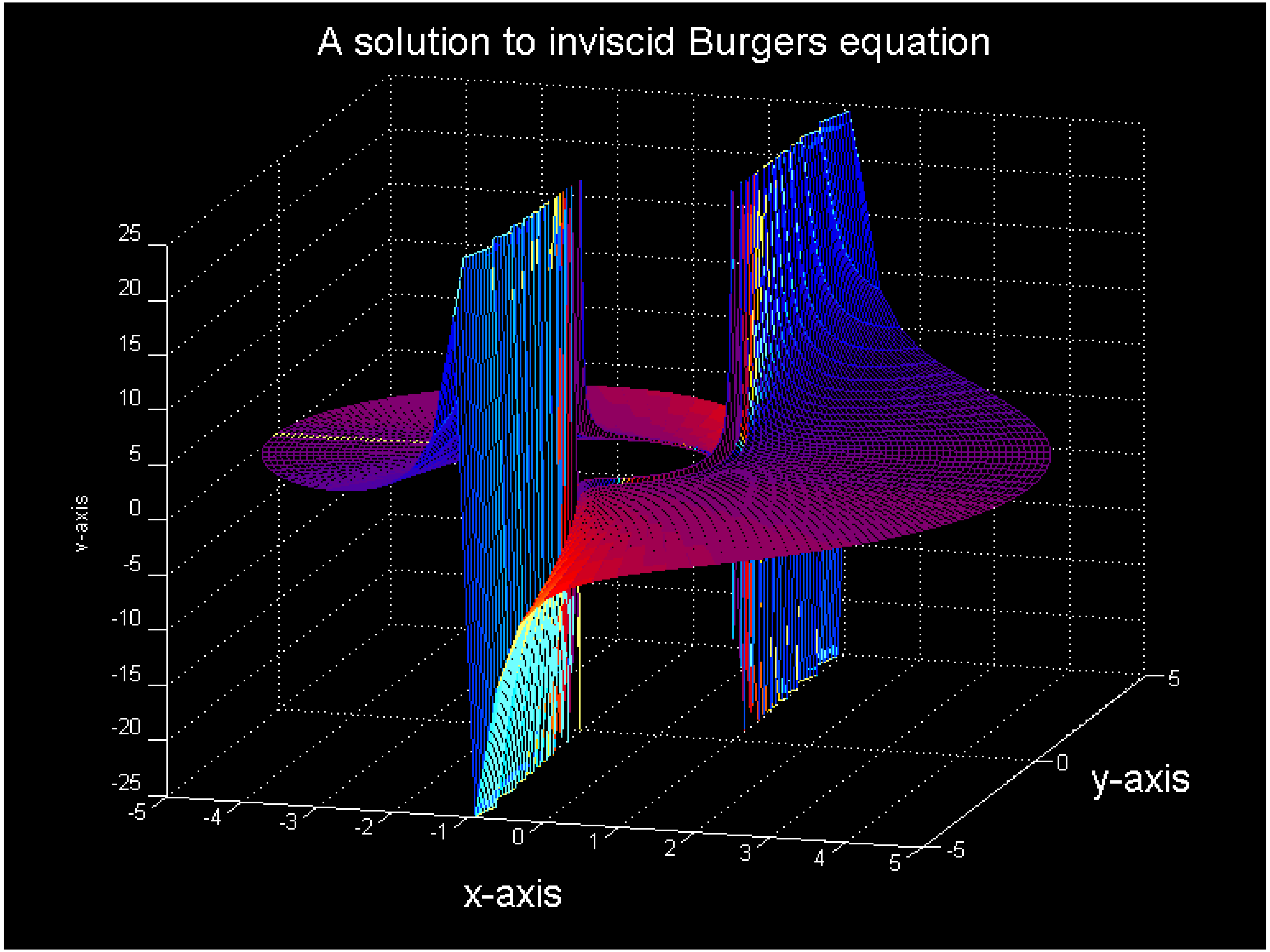
Here is a relevant pair

$$x = \rho\cos\omega, \quad y = \rho\sin\omega \quad (1 \le \rho < \infty, -\pi < \omega \le \pi),$$

$$w(x,y) = \arcsin\frac{1}{\rho} + \omega,$$

$$u(x,y) = \sqrt{\rho^2 - 1} + w(x,y), \quad v = \tan w.$$


A solution u to eikonal equation
10
8
6
4
2
0
-2
v-axis
5
0
-5
y-axis
-5
0
5
x-axis


A solution to inviscid Burgers equation
25
20
15
10
5
0
-5
-10
-15
-20
-25
y-axis
-5
-4
-3
-2
-1
0
1
2
3
4
5
x-axis
5
0
-5
y-axis

(vi) If ε is a positive parameter, the transformations attached to the following formulas

$$v_x = -\frac{1}{2\varepsilon} uv, \quad v_y = -\frac{1}{2}\left(u_x - \frac{1}{2\varepsilon} u^2\right)v,$$

$$u = -2\varepsilon \frac{v_x}{v},$$

(known as ***Cole-Hopf transformations***) are inverse of one another. They map solutions to

$$u_y + u u_x = \varepsilon \cdot u_{xx}$$

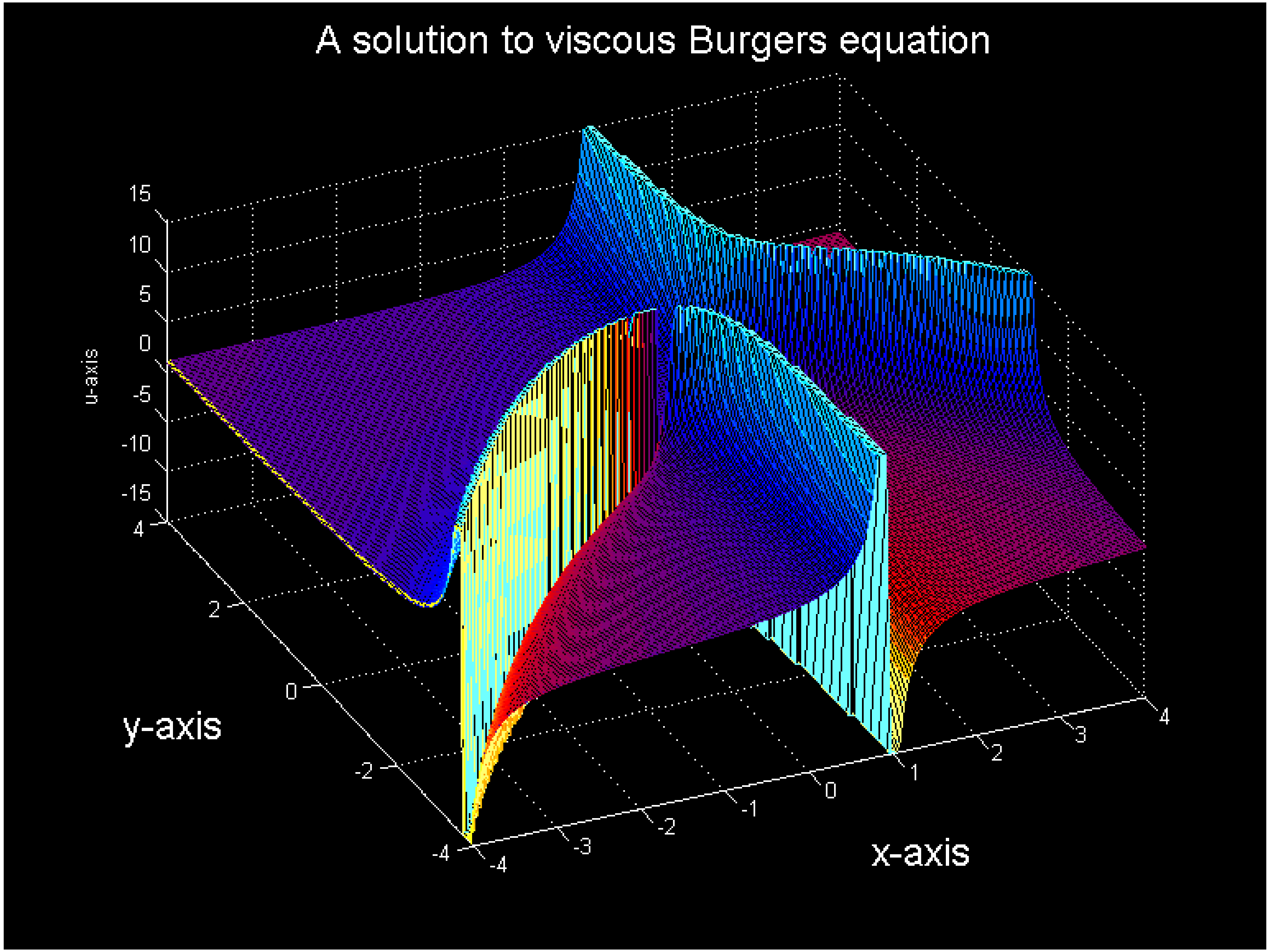
(***viscous Burgers equation***) into solutions to

$$v_y = \varepsilon \cdot v_{xx}$$

(***heat equation***), and viceversa.

Here is a sample pair

$$u(x,y) = -6\varepsilon \frac{x(x-2) + 2\varepsilon y}{x^2(x-3) + 6\varepsilon(x-1)y}, \quad v(x,y) = x^2(x-3) + 6\varepsilon(x-1)y.$$


A solution to viscous Burgers equation
15
10
5
0
-5
-10
-15
u-axis
4
2
0
-2
-4
y-axis
-4
-3
-2
-1
0
1
2
3
4
x-axis

(vii) If ε is a positive parameter, the transformation attached to the following system

$$v_x = u_x + 2\varepsilon \sin\left(\frac{v+u}{2}\right), \quad v_y = -u_y + \frac{2}{\varepsilon}\sin\left(\frac{v-u}{2}\right),$$

maps any solution to

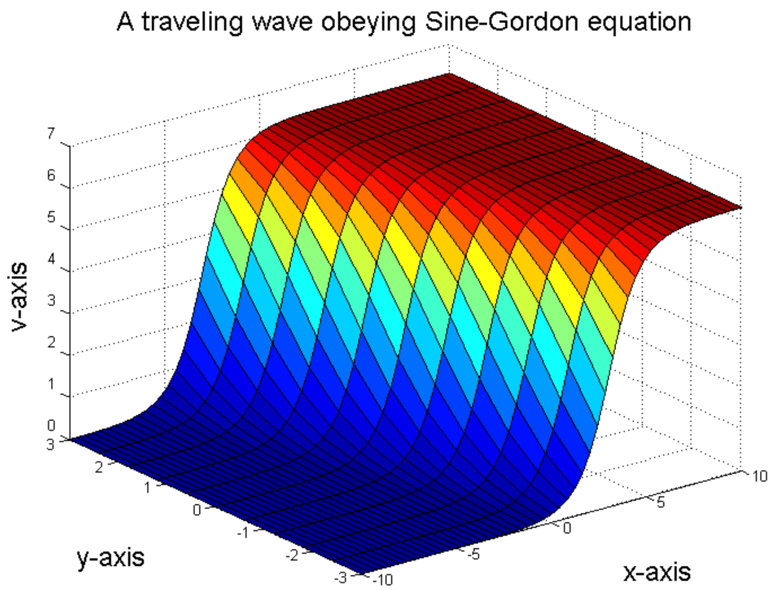
$$u_{xy} = \sin u$$

(known as ***Sine-Gordon equation***) into a solution to the same equation

$$v_{xy} = \sin v .$$

Here is a sample pair

$$u = 0, \quad \underbrace{v = 4\arctan\left(\exp\left(\varepsilon x + y/\varepsilon\right)\right)}_{\text{a traveling wave}}.$$



(viii) If p is a constant parameter such that $1 < p < \infty$, the transformations attached to these systems

$$\nabla v = \left|\nabla u\right|^{p-2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u, \quad \nabla u = \left|\nabla u\right|^{p/(p-1)-2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla v,$$

are the inverse of one another. They map a solution u to

$$\left[(p-1)u_x^2 + u_y^2\right] u_{xx} + 2(p-2) u_x u_y u_{xy} + \left[u_x^2 + (p-1)u_y^2\right] u_{yy} = 0$$

(the ***p-Laplace equation***) into a solution v to

$$\left[v_x^2+(p-1)v_y^2\right]v_{xx}-2(p-2)v_xv_y\,v_{xy}+\left[(p-1)v_x^2+v_y^2\right]v_{yy}=0$$

(the ***p/(p–1)-Laplace equation***) in such a way that

$$u_xv_x+u_yv_y=0.$$

(viii) Consider a two-dimensional, steady, adiabatic, isentropic, irrotational ***flow of a perfect gas***. The following equations and inequalities hold in proper units. First,

$$(\text{Pressure})\times(\text{Density})^{-\gamma}=1$$

— here ***γ = adiabatic constant***, $\gamma = 5/3$ for a monatomic gas, $\gamma = 7/5$ for a diatomic gas such as nitrogen or oxygen. Second, the ***velocity potential*** φ obeys both the inequality

$$1-\frac{\gamma-1}{2}|\nabla\varphi|^2\geq 0$$

and the equation

$$\left(1-\frac{\gamma+1}{2}\varphi_x^2-\frac{\gamma-1}{2}\varphi_y^2\right)\varphi_{xx}-2\varphi_x\varphi_y\,\varphi_{xy}+\left(1-\frac{\gamma-1}{2}\varphi_x^2-\frac{\gamma+1}{2}\varphi_y^2\right)\varphi_{yy}=0$$

— in other words, it is an extremal of the variational integral

$$\iint \left(1-\frac{\gamma-1}{2}\left|\nabla\varphi\right|^2\right)^{\gamma/(\gamma-1)} dxdy.$$

Third, the Bäcklund transform of φ given by

$$\nabla\psi = \left(1-\frac{\gamma-1}{2}\left|\nabla\varphi\right|^2\right)^{1/(\gamma-1)} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla\varphi$$

is a ***stream function*** — i.e. a function whose lines of steepest descent are orthogonal to equipotential lines.

Though not physically significant, the case where the adiabatic constant equals –1 is instrumental in the present context.

8. ELLIPTIC SOLUTIONS

8.1. **Formulas**. Elliptic solutions to **(B)** are the same as ***space-like maximal surfaces*** in the three-dimensional Minkowski space. Therefore, they can be parametrically represented by Kobayashi formulas [Ko]:

$$\lambda, \mu = \text{real parameters},$$

$$x = \mathrm{Re}\,\frac{1}{2}\int^{\lambda+i\mu} f(\zeta)\left[1+g(\zeta)^2\right]d\zeta,$$

$$y = \mathrm{Re}\,\frac{i}{2}\int^{\lambda+i\mu} f(\zeta)\left[1-g(\zeta)^2\right]d\zeta,$$

$$u = -\mathrm{Re}\int^{\lambda+i\mu} f(\zeta)g(\zeta)\,d\zeta \qquad (8.1)$$

— here $i=\sqrt{-1}$, f is holomorphic, g is a meromorphic function such that $|g|\neq 1$ and fg^2 is holomorphic.

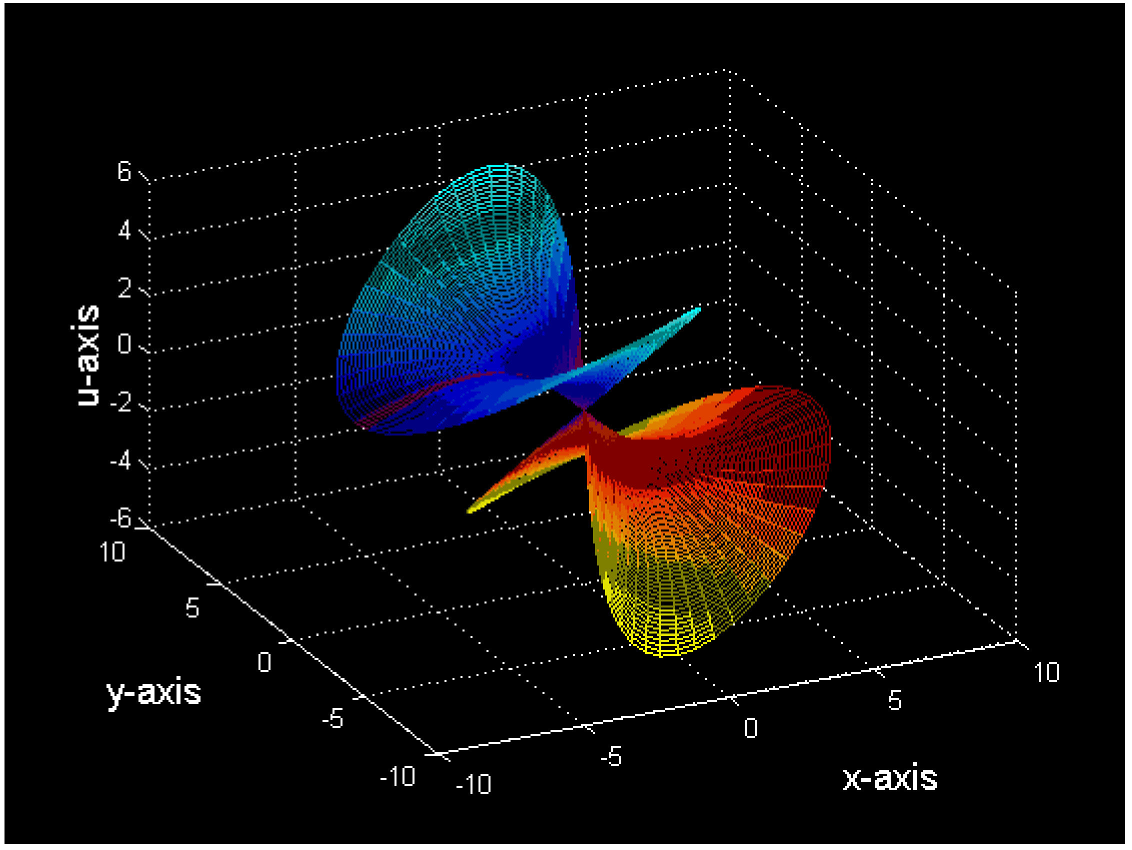
For example, putting

$$f(\zeta) = 6(1+\zeta)^2 \text{ and } g(\zeta) = (1-\zeta)/(1+\zeta)$$

and manipulating result in the quartic equation

$$(x-u)^4 = 27(x^2 + y^2 - u^2),$$

which supplies an elliptic solution to **(B)** and whose graph is shown below.



8.2. Allied minimal surfaces.

►Elliptic solutions to **(B)** are in one-to-one correspondence with ***standard minimal surfaces*** from Euclidean three-dimensional space.

►The ***Bäcklund transformations*** attached to the following formulas

$$\nabla v = \frac{1}{\sqrt{1-|\nabla u|^2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\cdot\nabla u\,, \quad \nabla u = \frac{1}{\sqrt{1+|\nabla v|^2}}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\cdot\nabla v\,, \tag{8.2}$$

amount to rotating gradients by ninety degrees, then stretching them suitably. They are the inverse of one another, and enjoy the following properties.

- The former acts on elliptic solutions u to equation **(B)** — such solutions are precisely what make

$$\frac{1}{\sqrt{1-|\nabla u|^2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\cdot\nabla u$$

  well-defined and a gradient.

- The latter acts on solutions v to the following equation

$$(1+v_y^2)v_{xx}-2v_yv_xv_{xy}+(1+v_x^2)v_{yy}=0\,, \tag{8.3}$$

i.e. on functions whose graphs are ***minimal surfaces*** — such functions are precisely what make

$$\frac{1}{\sqrt{1+\left|\nabla v\right|^2}}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\cdot\nabla v$$

a gradient.

- Both convert any elliptic solution to **(B)** into a non-parametric minimal surface, and any non-parametric minimal surface from Euclidean three-dimensional space into an elliptic solution to **(B)**.

►The Bäcklund transformations in hand allow properties of equation **(B)** to conveniently follow from properties of minimal surfaces. For instance, Bernstein-Calabi theorem on elliptic entire solutions to **(B)** can be immediately demonstrated along this line.

►They also call the notion of ***Chaplygin gas*** into play. According to usage, “Chaplygin gas” is a nickname for a hypothetical fluid whose adiabatic constant equals (–1), i.e. whose density and pressure are inversely proportional to one another. If units are appropriate,

$$\gamma = -1,$$

and the space dimension is two, the equations

$$\left(1-\frac{\gamma+1}{2}\varphi_x^2-\frac{\gamma-1}{2}\varphi_y^2\right)\varphi_{xx}-2\varphi_x\varphi_y\,\varphi_{xy}+\left(1-\frac{\gamma-1}{2}\varphi_x^2-\frac{\gamma+1}{2}\varphi_y^2\right)\varphi_{yy}=0$$

$$\nabla\psi=\left(1-\frac{\gamma-1}{2}\left|\nabla\varphi\right|^2\right)^{1/(\gamma-1)}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\nabla\varphi$$

govern the a velocity potential φ and a stream function ψ of such a fluid. The following assertions result. First: the Bäcklund transformations attached to equations (8.2) are precisely those relating ***a velocity potential and a stream function*** of a Chaplygin gas. Second: while the minimal surface equation governs a velocity potential, ***equation (B) governs a stream function*** of a Chaplygin gas.

►Either one of equations (8.2) implies

$$\left(1-u_x^2-u_y^2\right)\cdot\left(1+v_x^2+v_y^2\right)=1, \quad u_x v_x+u_y v_y=0, \tag{8.4}$$

a first-order fully non-linear partial differential system having a rotation-invariant structure.

System (8.4) plus the condition

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$$

imply both equations (8.2).

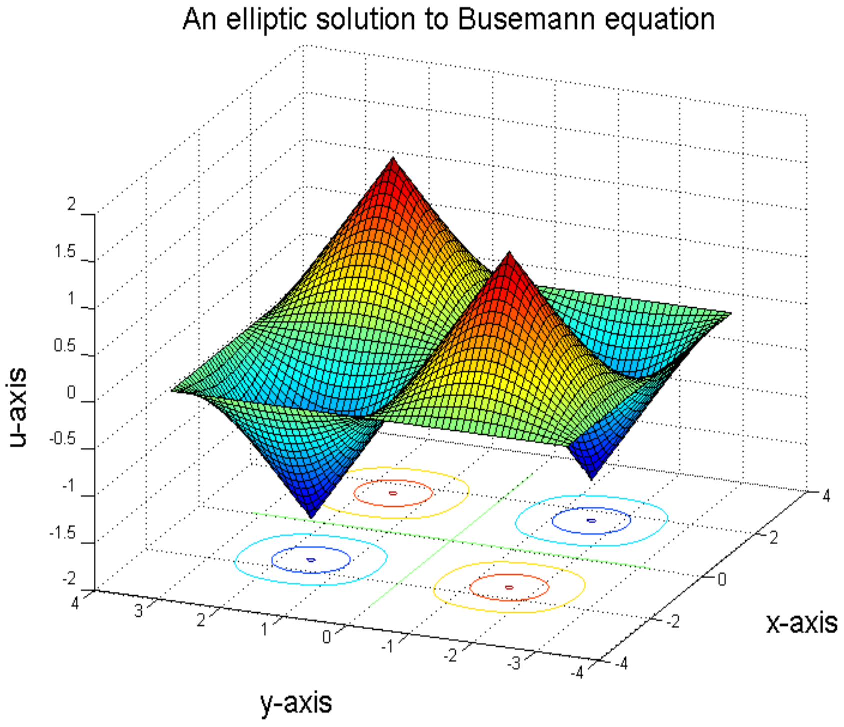
The ***entries*** of any solution pair to (8.4), whose Jacobian determinant does not change its sign, satisfy equation **(B)** and represent a minimal surface, respectively.

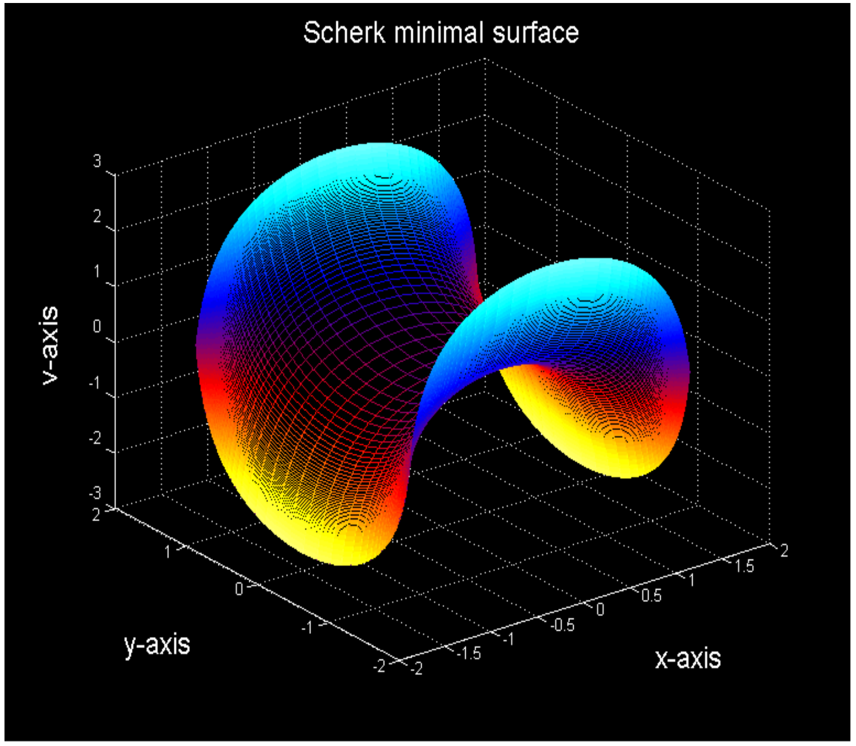
System (8.4) ***pairs off*** solutions to equation **(B)** and minimal surfaces, whereas Bäcklund transformations (8.2) ***decouple*** solution pairs to (8.4).

Here is a sample pair

$$u(x,y) = \arcsin(\sin x \cdot \sin y)\,, \quad v(x,y) = \log\left(\frac{\cos x}{\cos y}\right)$$

— the former entry is an elliptic solution to equation **(B)**, the latter is ***Scherk's minimal surface***.





9. Hyperbolic solutions

9.1. Formulas. ***Hyperbolic*** solutions to equation **(B)** result from the formulas of Gu and Li:

$$\lambda, \mu = \text{parameters},$$

$$x = \int^{\lambda} a(\lambda) \cos \lambda \, d\lambda + \int^{\mu} b(\mu) \cos \mu \, d\mu,$$

$$y = \int^{\lambda} a(\lambda) \sin \lambda \, d\lambda + \int^{\mu} b(\mu) \sin \mu \, d\mu,$$

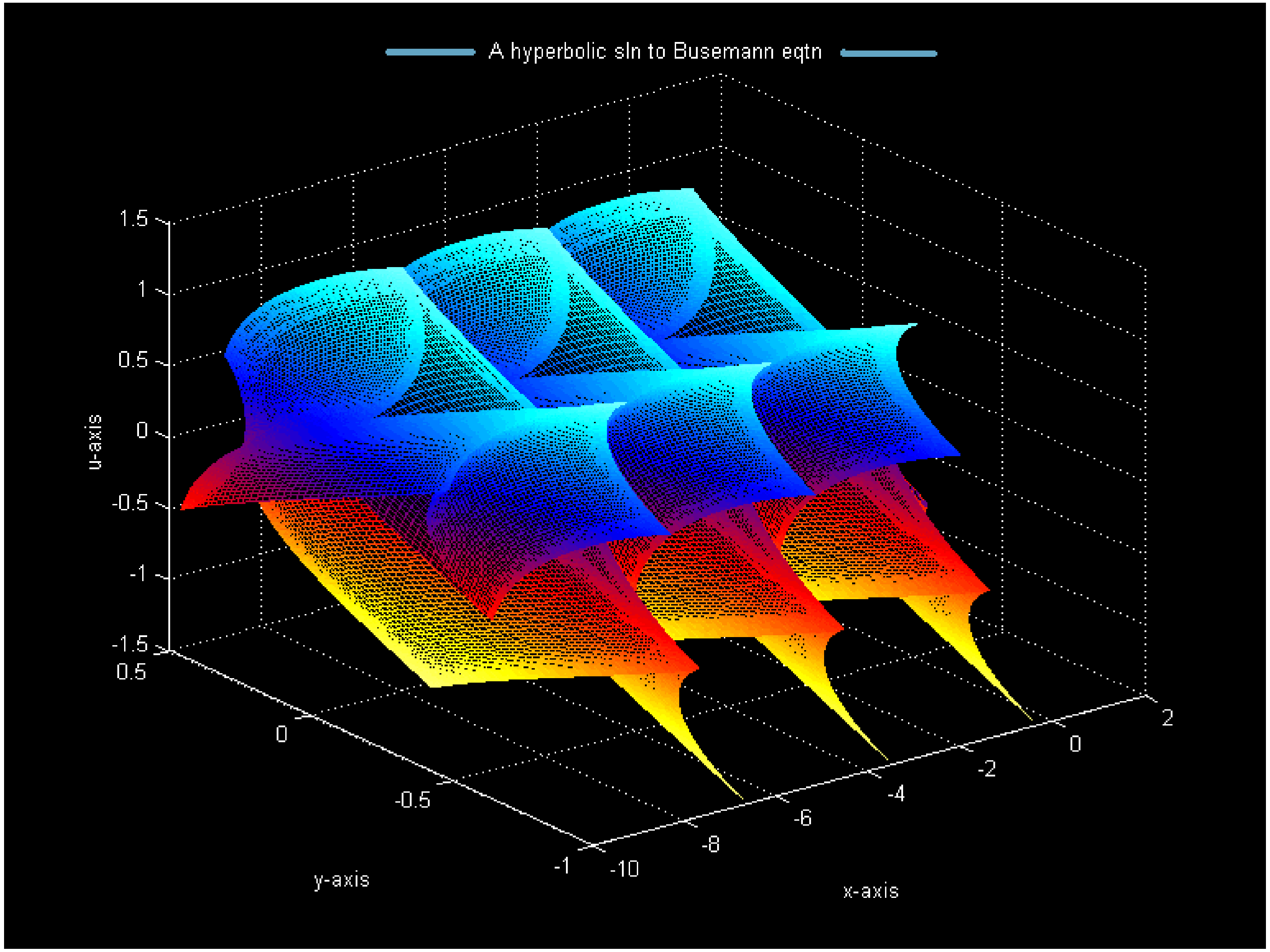
$$u = \int^{\lambda} a(\lambda) d\lambda + \int^{\mu} b(\mu) d\mu, \tag{9.1}$$

which involve two non-zero real functions *a* and *b* at user's disposal, and imply

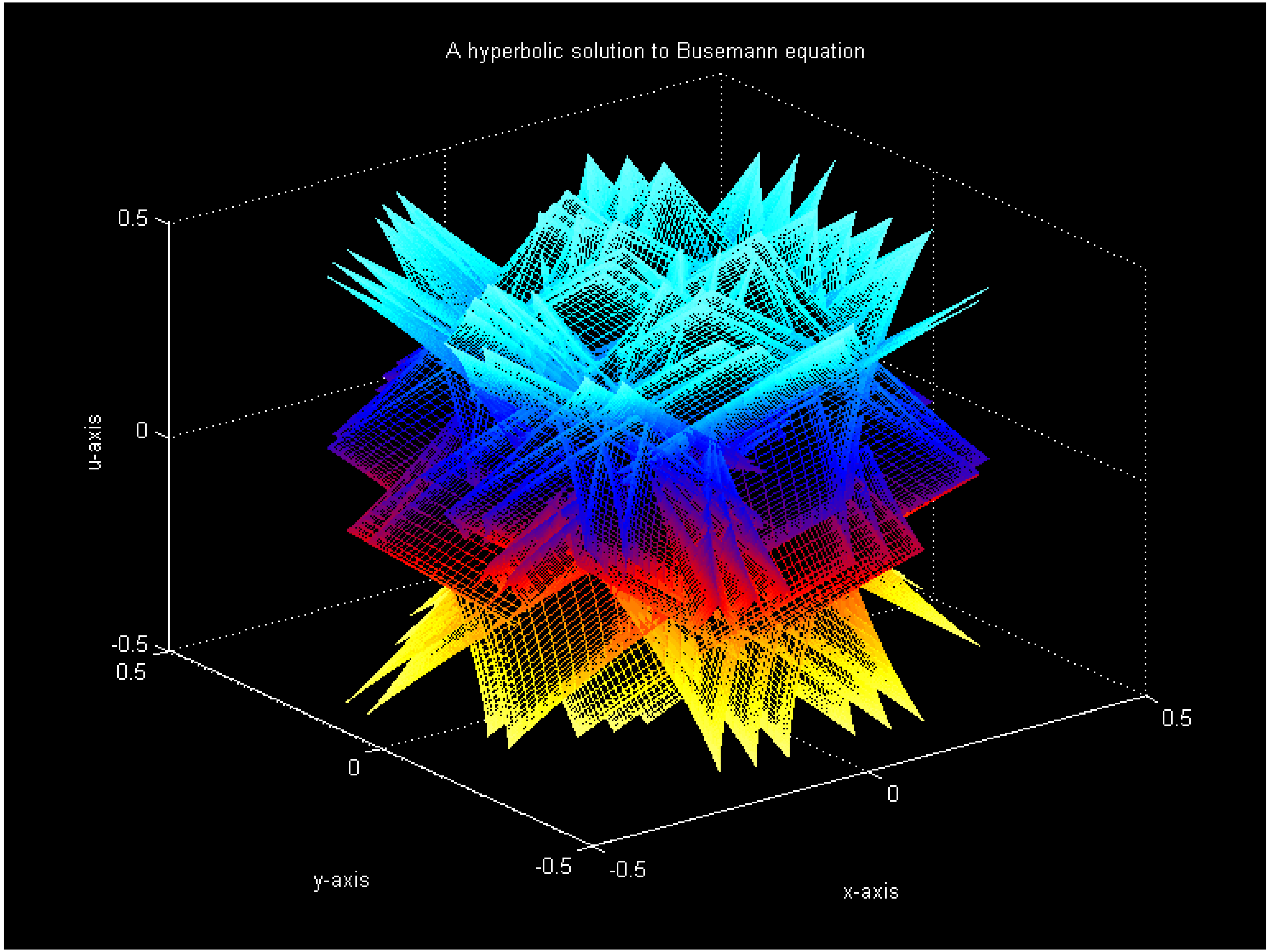
$$\cos\left(\frac{\lambda-\mu}{2}\right) \cdot u_x = \cos\left(\frac{\lambda+\mu}{2}\right), \quad \cos\left(\frac{\lambda-\mu}{2}\right) \cdot u_y = \sin\left(\frac{\lambda+\mu}{2}\right).$$

Here are examples.

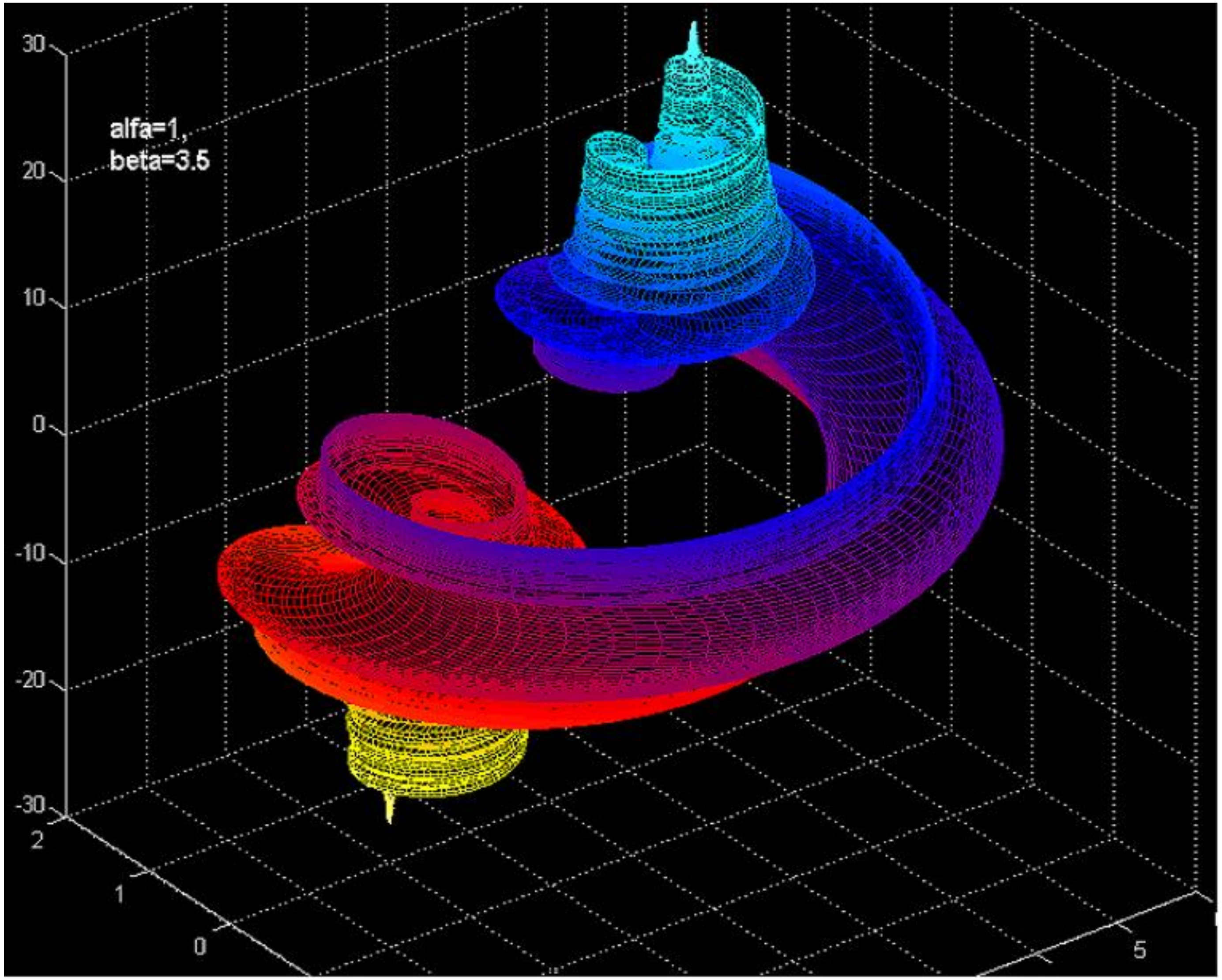
$$a(\lambda) = \cos(2\lambda)\,, \quad b(\mu) = \cos\mu\,;$$



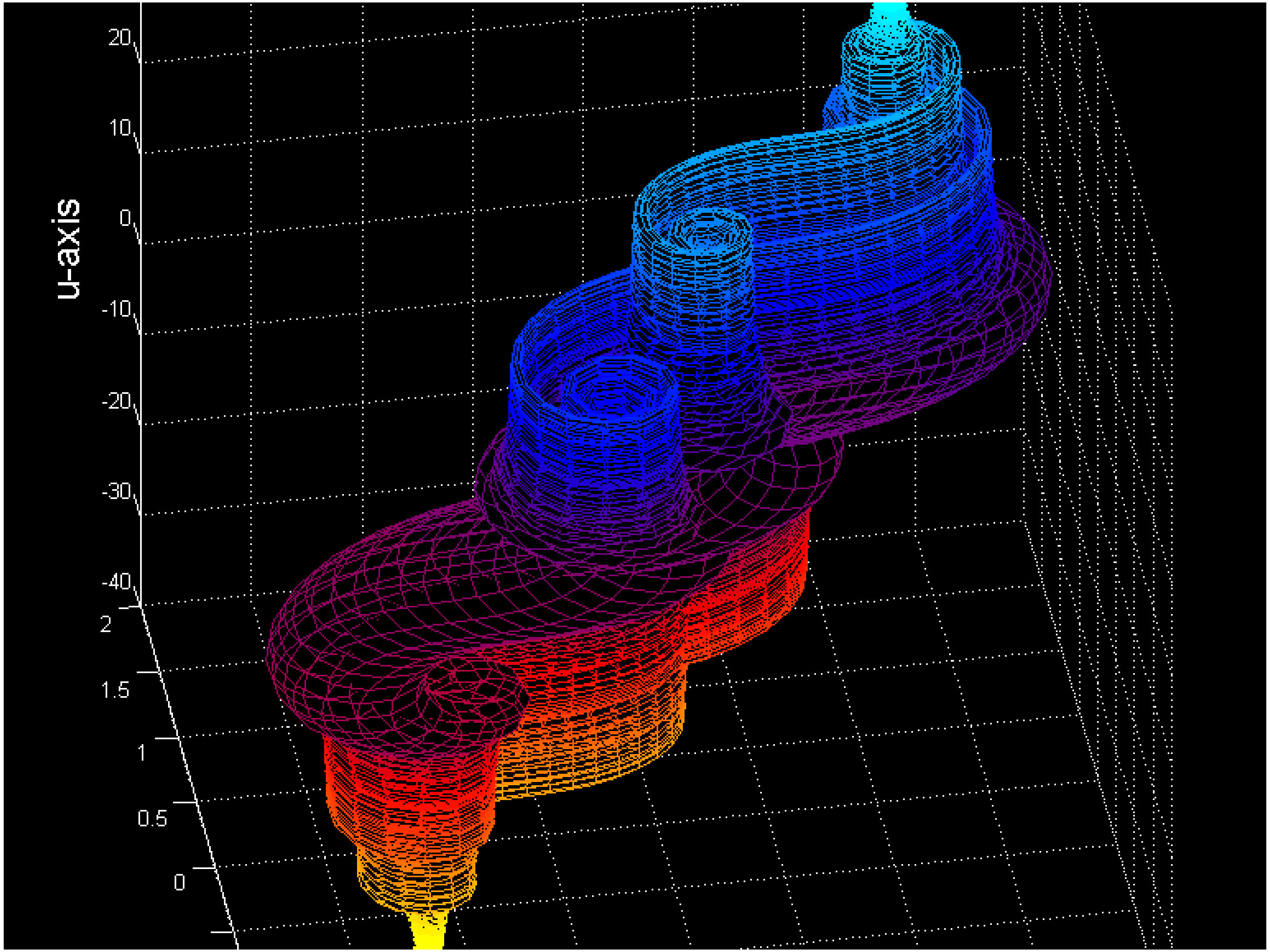
$$a(\lambda) = \cos(5\lambda)\,, \quad b(\mu) = \sin(\sqrt{15}\mu)\,;$$



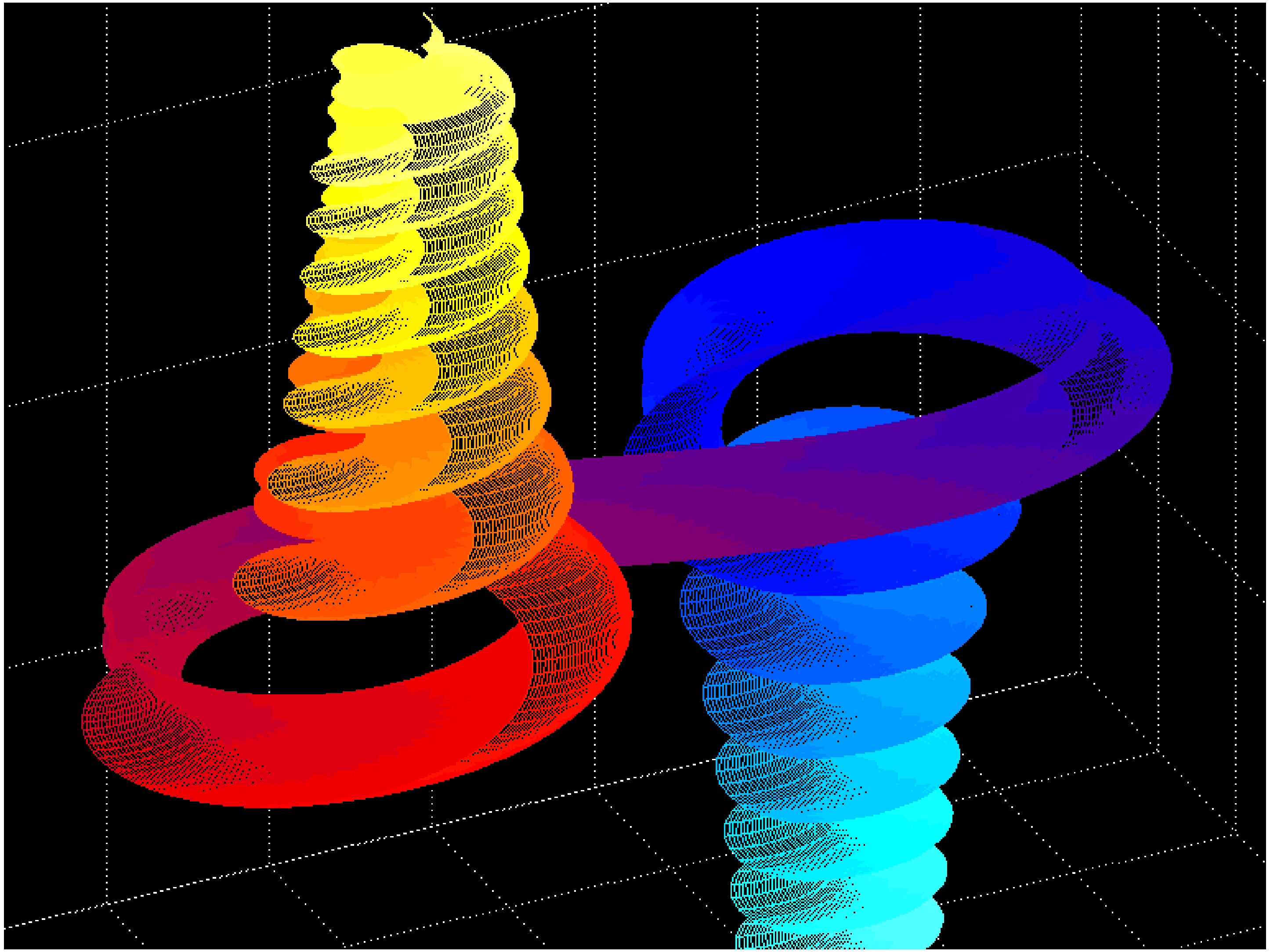
$$a(\lambda)=\alpha\Big/\sqrt{1+\lambda^2}\ ,\quad b(\mu)=\beta\Big/\sqrt{1+\mu^2}\ ;$$



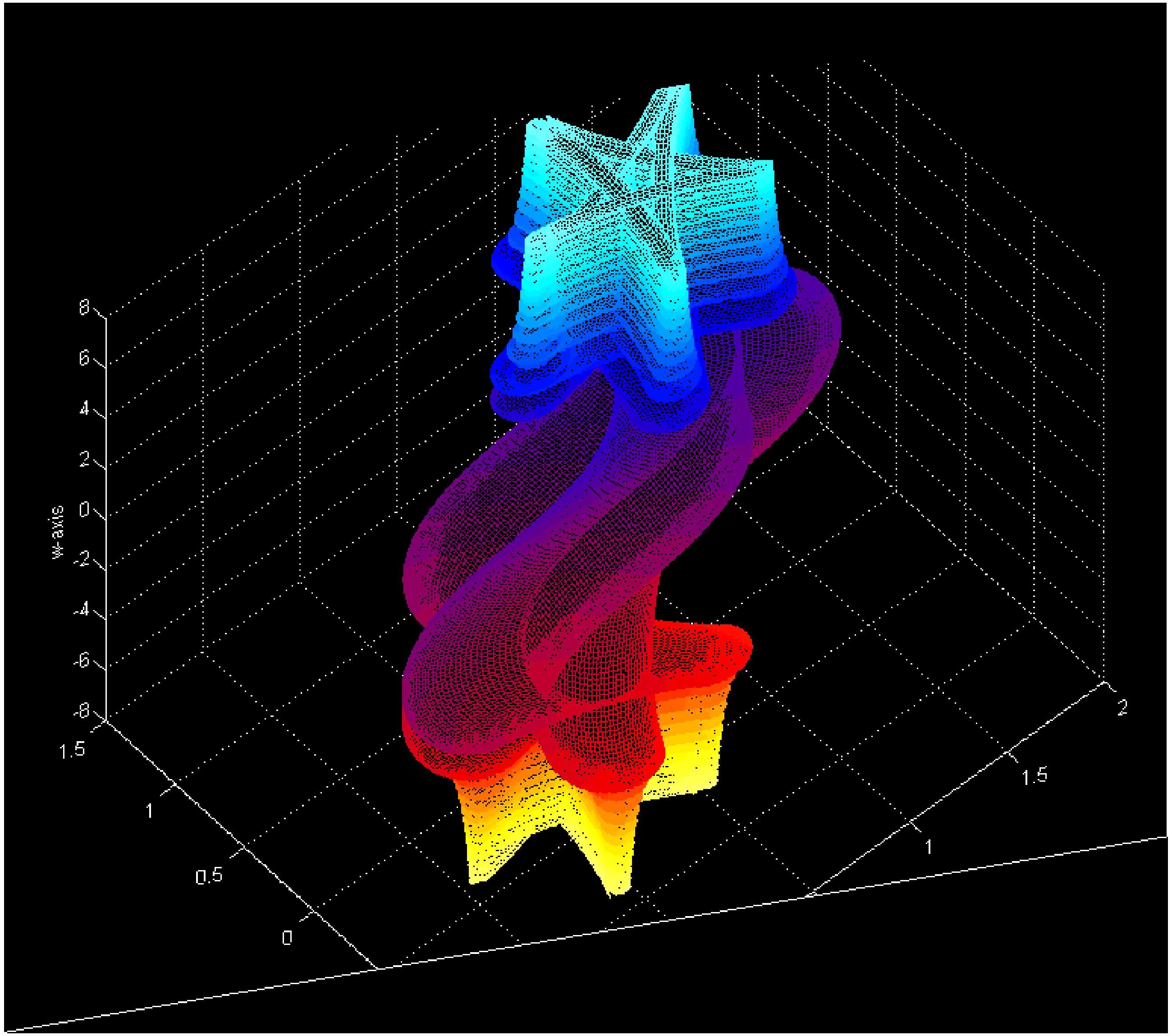
$$a(\lambda) = \alpha \big/ \sqrt{2\pi\lambda}\ , \quad b(\mu) = \beta \big/ \sqrt{2\pi\mu}\ ;$$



$$a(\lambda) = \alpha \frac{\sin(\lambda/5)}{\lambda}, \quad b(\mu) = \beta\Big/\sqrt{2\pi\mu}\ ;$$



$$a(\lambda) = \alpha \sin(5\lambda)\,, \quad b(\mu) = \beta \big/ \sqrt{2\pi\mu}\,.$$



9.2. Allied solutions.

►The Bäcklund transformations attached to the following formulas

$$\nabla v = \frac{1}{\sqrt{|\nabla u|^2 - 1}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u, \quad \nabla u = \frac{1}{\sqrt{|\nabla v|^2 - 1}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla v, \quad (9.2)$$

are inverse of one another, and enjoy the following properties.

- Both act on hyperbolic solutions to **(B)**.
- They convert any hyperbolic solution u to equation **(B)** into another hyperbolic solution v to the same equation.
- They imply

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \geq 2$$

— in particular, make the mapping

$$[x \quad y] \mapsto [u(x,y) \quad v(x,y)]$$

a local one-to-one diffeomorphism.

►Either one of equations (9.2) implies

$$\left(u_x^2+u_y^2-1\right)\cdot\left(v_x^2+v_y^2-1\right)=1, \quad u_x v_x+u_y v_y=0, \tag{9.3}$$

a first-order fully non-linear partial differential system endowed with a rotation-invariant structure.

System (9.3) plus the condition

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} > 0$$

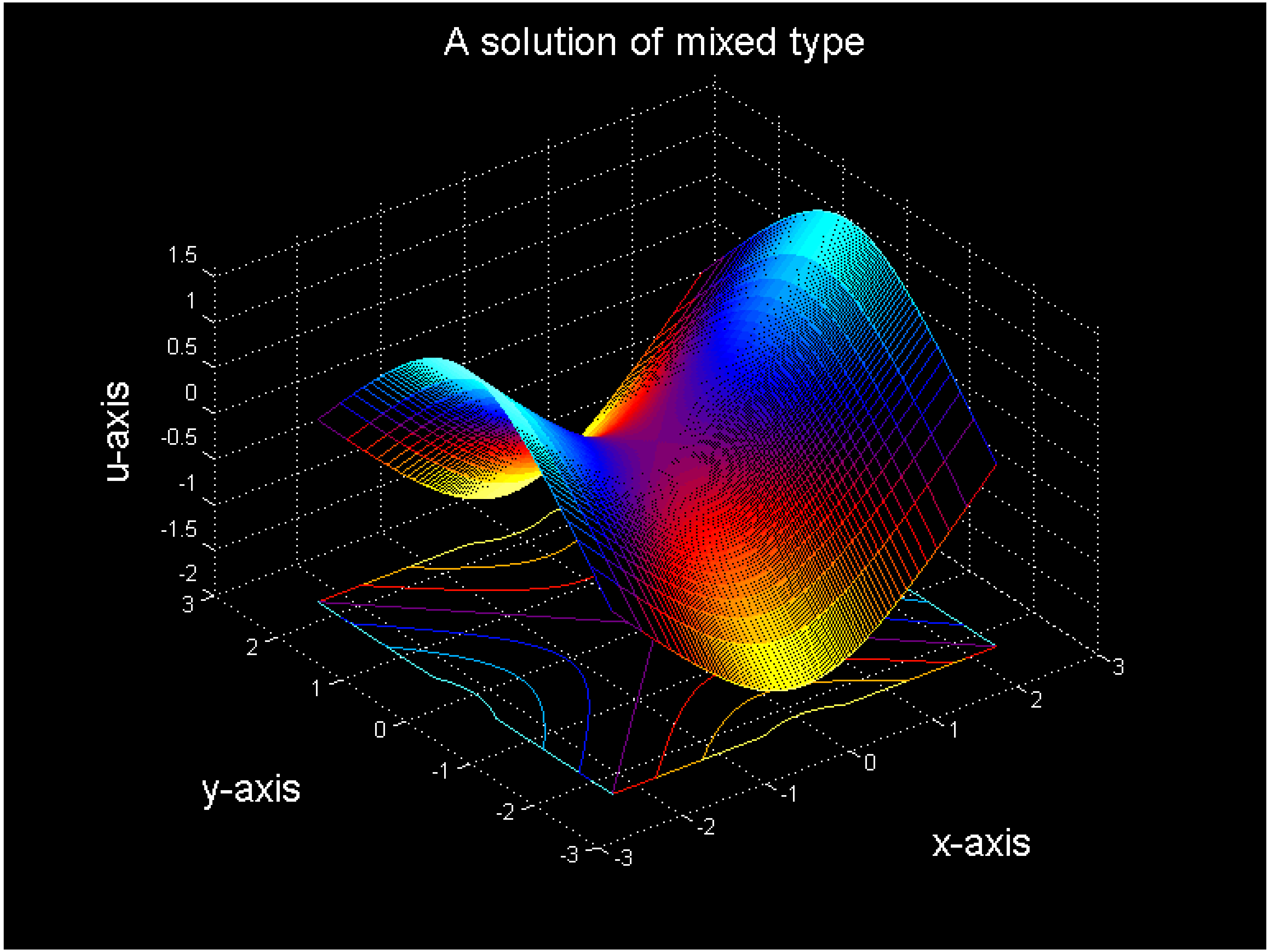
imply equations (9.2) simultaneously.

Both entries of any solution pair to (9.3), whose Jacobian determinant does not change its sign, satisfy **(B)** — in other words, system (9.3) ***pairs off*** solutions to equation **(B)**.

Equations (9.2) ***decouple*** all solution pairs to (9.3) whose Jacobian determinant is positive.

Example. The following solution to **(B)**

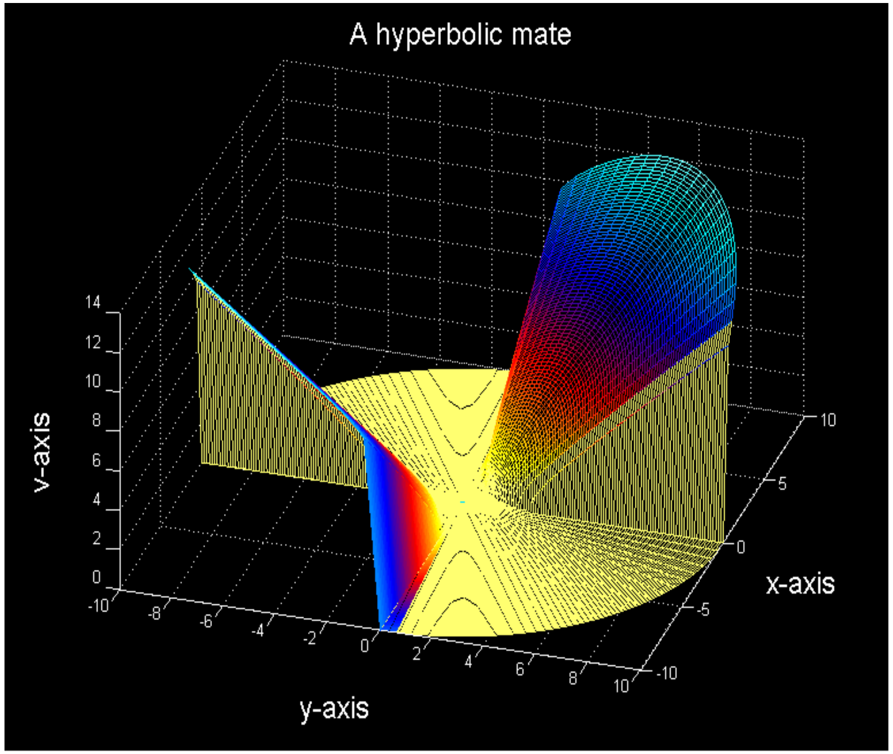
$$u(x,y)=\log\left(\frac{\cosh x}{\cosh y}\right),$$

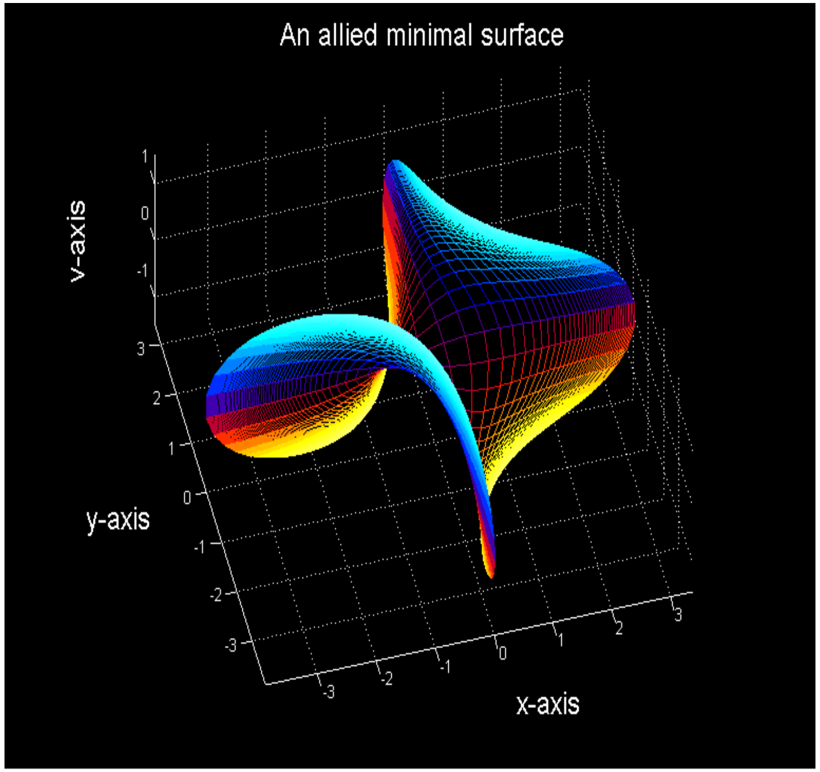

A solution of mixed type
u-axis
y-axis
x-axis
1.5
1
0.5
0
-0.5
-1
-1.5
-2
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

is of mixed hyperbolic-elliptic type and has the following two mates

$$v(x,y) = \operatorname{arccosh}(\sinh x \cdot \sinh y), \quad \underbrace{v(x,y) = \arcsin(\sinh x \cdot \sinh y)}_{\text{a minimal surface}}$$

— the former is a hyperbolic solution to **(B),** the latter is a minimal surface.





7.3. D'Alembert equation. Suppose u and v are ***allied hyperbolic solutions*** to equation **(B)**, i.e.

$$\nabla v = \frac{1}{\sqrt{|\nabla u|^2 - 1}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u, \quad \nabla u = \frac{1}{\sqrt{|\nabla v|^2 - 1}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla v.$$

Think of u and v as ***curvilinear coordinates***, and think of ***x and y as functions of u and v*** — in other words, interchange the role of dependent and independent variables.

The following properties hold.

- x and y obey both

$$\frac{\partial}{\partial v}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\sqrt{\frac{1}{x_u^2 + y_u^2} - 1}}_{\text{stretching}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation}} \frac{\partial}{\partial u}\begin{bmatrix} x \\ y \end{bmatrix}, \tag{9.4}$$

and

$$\underbrace{\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2}_{E}+\underbrace{\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2}_{G}=1, \quad \underbrace{\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}}_{F}=0. \qquad (9.5)$$

Observe that letting *E*, *F* and *G* stand for the coefficients of the Euclidean metric, i.e. satisfy

$$(dx)^2+(dy)^2=E(du)^2+2Fdudv+G(dv)^2,$$

allows system (9.5) to read thus

$$E+G=1, \quad F=0.$$

- *x* and *y* obey ***D'Alembert equation***, i.e.

$$\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)\begin{bmatrix} x \\ y \end{bmatrix}=0, \qquad (9.6)$$

Proof number 1. Coupling

$$\nabla v = \frac{1}{\sqrt{|\nabla u|^2 - 1}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla u$$

and

$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}^{-1}$$

results in

$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix} = (u_x^2 + u_y^2)^{-1} \begin{bmatrix} u_x & -u_y \\ u_y & u_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{u_x^2 + u_y^2 - 1} \end{bmatrix}.$$

The claim follows.

Proof number 2. System (9.4) can be recast as

$$\frac{\partial \mathbf{w}}{\partial v} = \mathbf{f}\left(\frac{\partial \mathbf{w}}{\partial u}\right),$$

provided the following notations and abbreviations are in use

$$\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix},$$

$$\rho^2 = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2, \quad g(\rho) = \sqrt{\frac{1}{\rho^2} - 1}, \quad \mathbf{f}\left(\frac{\partial \mathbf{w}}{\partial u}\right) = g(\rho)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\frac{\partial \mathbf{w}}{\partial u}.$$

We consequently have

$$\frac{\partial^2 \mathbf{w}}{\partial v^2} = \left[\mathbf{f}'\left(\frac{\partial \mathbf{w}}{\partial u}\right)\right]^2 \frac{\partial^2 \mathbf{w}}{\partial u^2},$$

$$\mathbf{f}'\left(\frac{\partial \mathbf{w}}{\partial u}\right)=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\left\{g(\rho)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\frac{1}{\rho}\frac{dg(\rho)}{d\rho}\begin{bmatrix}\left(\frac{\partial x}{\partial u}\right)^2 & \frac{\partial x}{\partial u}\frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial u}\frac{\partial y}{\partial u} & \left(\frac{\partial y}{\partial u}\right)^2\end{bmatrix}\right\},$$

$$\left[\mathbf{f}'\left(\frac{\partial \mathbf{w}}{\partial u}\right)\right]^2=-\left[\rho g(\rho)\frac{dg(\rho)}{d\rho}+g(\rho)^2\right]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$\rho g(\rho)\frac{dg(\rho)}{d\rho}+g(\rho)^2=-1.$$

We conclude

$$\left(\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}\right)\begin{bmatrix}x\\ y\end{bmatrix}=0,$$

as claimed.

The above statements yield the following consequence.

►*The* ***hyperbolic*** *solutions u and v to equation* **(B)**, *which are* ***paired*** *by Bäcklund transformations (9.2), can be represented thus*

$$u=(\lambda+\mu)/\sqrt{2}\,,\qquad v=(\lambda-\mu)/\sqrt{2}\,,$$
$$x=\left[A(\lambda)+B(\mu)\right]/\sqrt{2}\,,\quad y=\left[C(\lambda)+D(\mu)\right]/\sqrt{2} \tag{9.7}$$

— here

$$\lambda,\mu=\text{parameters}$$

and A, B, C, D satisfy

$$\left[\frac{dA}{d\lambda}(\lambda)\right]^2+\left[\frac{dC}{d\lambda}(\lambda)\right]^2\equiv 1\,,\quad \left[\frac{dB}{d\mu}(\mu)\right]^2+\left[\frac{dD}{d\mu}(\mu)\right]^2\equiv 1\,.$$

10. Initial value problems

Let the following ingredients be in stock.

First,

$$\overbrace{x=\alpha(s), \quad y=\beta(s)}^{\text{initial curve}}$$

— a parametric representation of a smooth, plane, ***initial curve***. Suppose ***s* = arc length**.

Second,

$$s \mapsto \gamma(s),$$

— a real-valued smooth function.

An ***initial value problem*** consists in looking for a solution *u* to

$$\left(u_y^2-1\right)\cdot u_{xx}-2u_x u_y\cdot u_{xy}+\left(u_x^2-1\right)\cdot u_{yy}=0 \qquad \textbf{(B)}$$

that satisfies

$$u(x,y)=\gamma(s), \quad \underbrace{-u_x(x,y)\frac{d\beta}{ds}(s)+u_y(x,y)\frac{d\beta}{ds}(s)}_{\text{normal derivative of } u}=0$$

for all x and y that run on the initial curve.

Such a problem is ***non-characteristic*** if

$$\left|\frac{d\gamma}{ds}(s)\right|\neq 1 \quad \text{everywhere;} \tag{8.4}$$

if even

$$\left|\frac{d\gamma}{ds}(s)\right|>1 \quad \text{everywhere,} \tag{8.5}$$

then any solution leaves the initial curve in a ***hyperbolic status***.

Ad hoc formulas read as follows

$$\sqrt{2}u = \lambda + \mu\,, \qquad \overbrace{\sqrt{2}v}^{\text{allied solution}} = \lambda - \mu\,, \qquad \sqrt{2}x = A(\lambda) + B(\mu)\,, \qquad \sqrt{2}y = C(\lambda) + D(\mu)\,,$$

$$A'\left(\frac{\gamma(s)}{\sqrt{2}}\right) = \frac{1}{\gamma'(s)} \cdot \alpha'(s) \pm \sqrt{1 - \frac{1}{\gamma'(s)^2}} \cdot \beta'(s),$$

$$A\left(\frac{\gamma(s)}{\sqrt{2}}\right) + B\left(\frac{\gamma(s)}{\sqrt{2}}\right) = \sqrt{2}\,\alpha(s),$$

$$C'\left(\frac{\gamma(s)}{\sqrt{2}}\right) = \mp\sqrt{1 - \frac{1}{\gamma'(s)^2}} \cdot \alpha'(s) + \frac{1}{\gamma'(s)} \cdot \beta'(s),$$

$$C\left(\frac{\gamma(s)}{\sqrt{2}}\right) + D\left(\frac{\gamma(s)}{\sqrt{2}}\right) = \sqrt{2}\,\beta(s).$$

Here is an example:

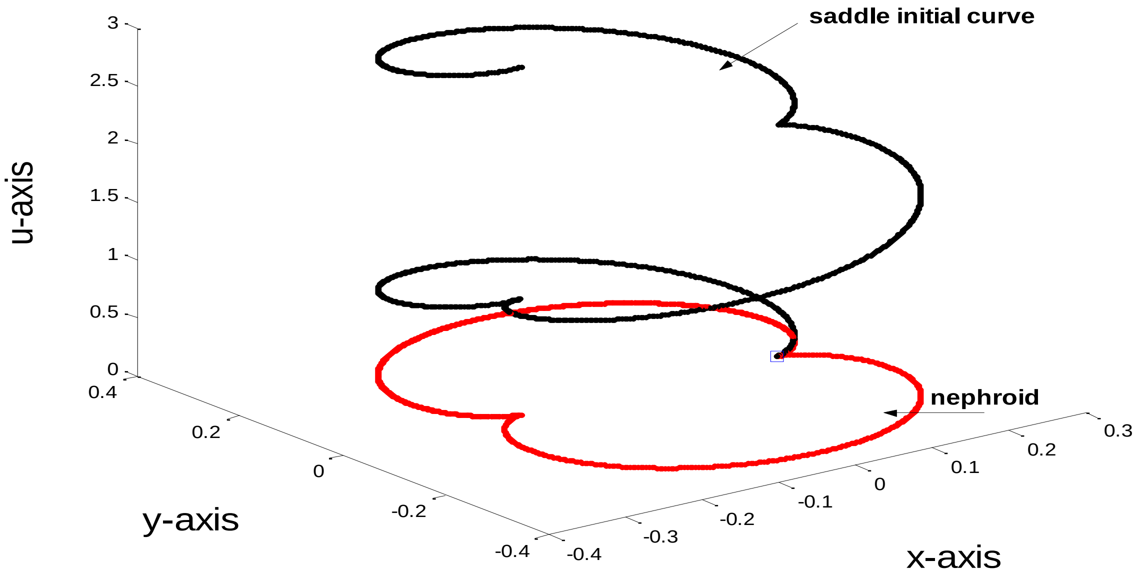
$$
\begin{aligned}
&s = \text{arc length}, \quad t = \text{another parameter},\\
&k = \operatorname{int}(s), \quad t = k\pi + \arccos(2k+1-2s),\\
&\alpha(s) = \tfrac{1}{12}(3\cos t - \cos 3t), \quad \beta(s) = \tfrac{1}{12}(3\sin t - \sin 3t)
\end{aligned}
$$

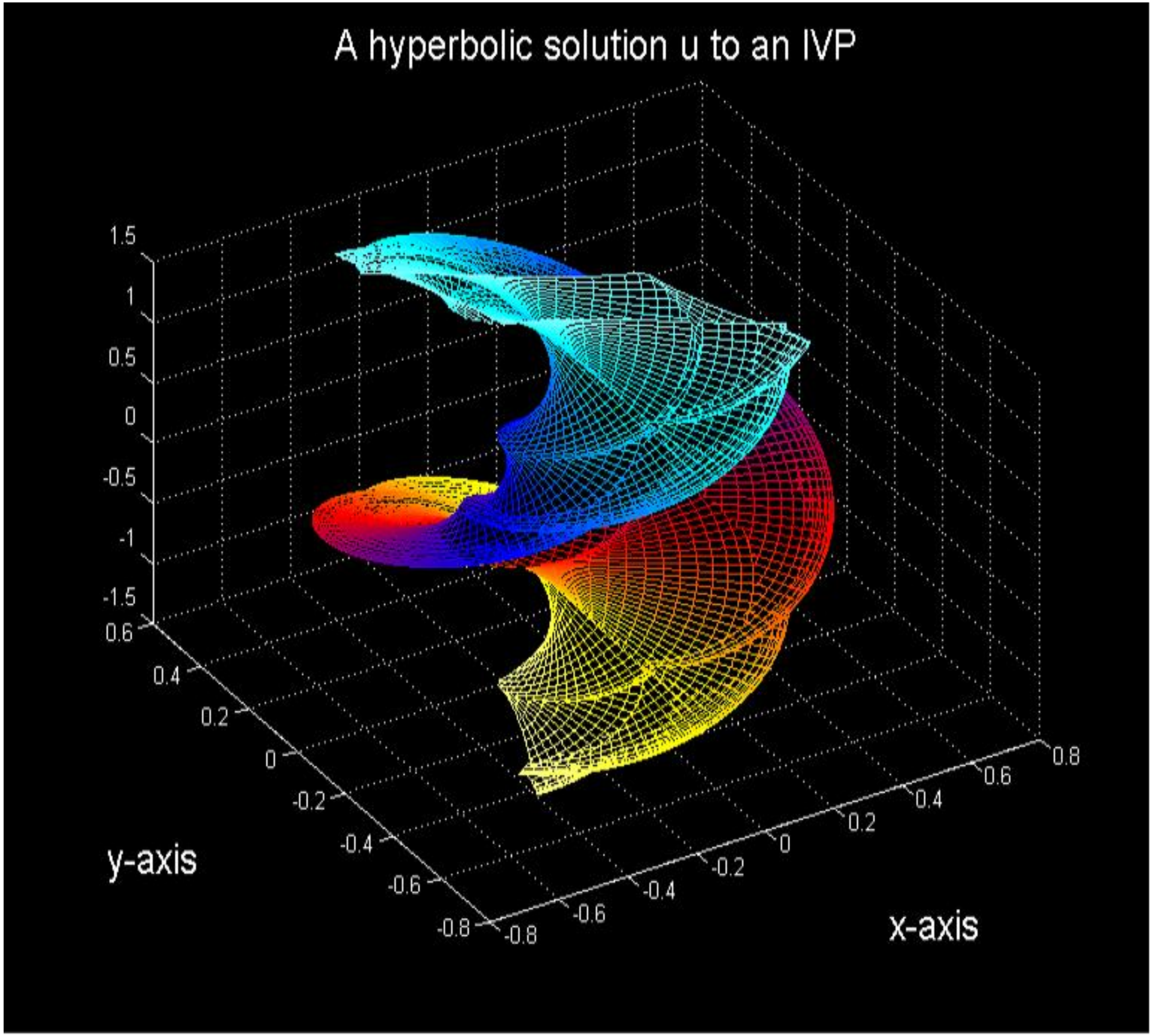
— a ***nephroid of Huygens*** — and

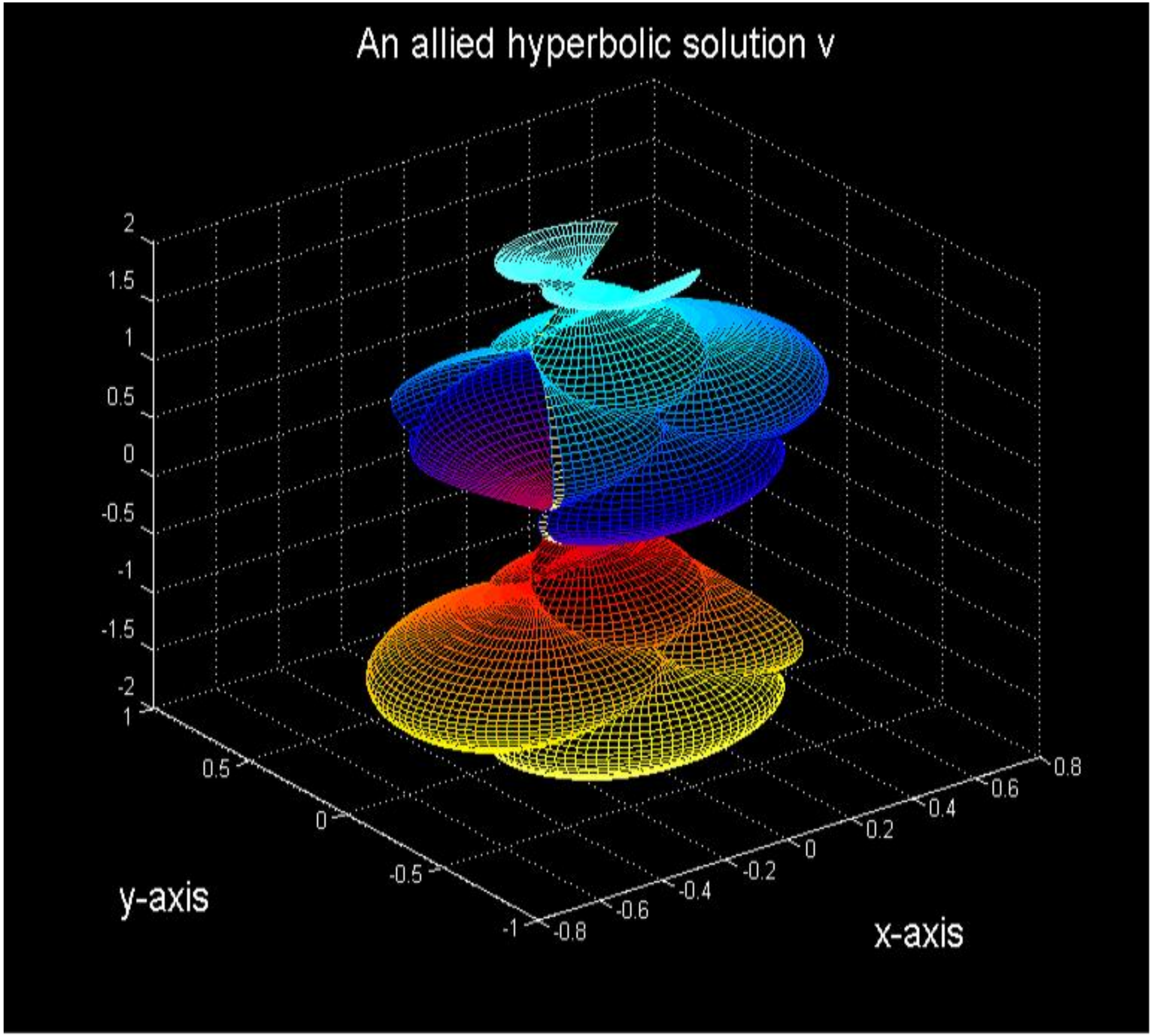
$$\gamma(s) = 2.0156 \cdot s$$

— a ***helix*** above the nephroid.

Views of the initial curve




A hyperbolic solution u to an IVP
1.5
1
0.5
0
-0.5
-1
-1.5
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
y-axis
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
x-axis


An allied hyperbolic solution v
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
1
0.5
0
-0.5
-1
y-axis
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
x-axis