Well-Structured

Parameterized Networks of Systems

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1st Workshop on Parameterized Verification, Roma, Sep. 6th, 2014

WSTS FOR PV?

- *Well-structured systems* (WSTS) are a family of infinite-state models supporting generic verification algorithms based on well-quasi-ordering (WQO) theory.
- WSTS invented in 1987, developed and popularized in 1996–2005 by Abdulla & Jonsson, Finkel & Schnoebelen, etc. First used with Petri nets/VASS extensions, channel systems, counter machines, integral automata, etc.
- Used in software verification, communication protocols, . . . In particular, for distributed algorithms, WSTS have been used for verification of parameterized networks. Useful for proving safety/for finding minimal unsafe start configurations.
- WSTS still thriving today, with several new models (based on wqos on graphs, etc.), or applications (deciding data logics, modal logics, etc.) proposed every year.
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Outline of the Talk

- Part 1: **Basics of WSTS.**

  Recalling the basic definition, with Broadcast protocols and Timed-arc nets as examples

- Part 2: **Verifying WSTS.**

  Two simple verification algorithms, deciding Termination and Coverability

- Part 3: **A few words on complexity.**

  Looking at controlled bad sequences and bounding their length

Part 1 What are WSTS?

What Are WSTS?

Def. A WSTS is an ordered TS $\mathcal{S} = (S, \to, \leqslant)$ that is monotonic and such that $(S, \leqslant)$ is a well-quasi-ordering (a wqo, more later).

Recall:

– transition system (TS): $\mathcal{S} = (S, \to)$ with steps e.g. "$s \to s'$"

– ordered TS: $\mathcal{S} = (S, \to, \leqslant)$ with smaller and larger states, e.g. $s \leqslant t$

– monotonic TS: ordered TS with
$\big(s_1 \to s_2 \text{ and } s_1 \leqslant t_1\big)$ implies $\exists t_2 \in S : \big(t_1 \to t_2 \text{ and } s_2 \leqslant t_2\big)$,
i.e., "larger states simulate smaller states".

Equivalently: $\leqslant$ is a wqo and a simulation.

NB. Starting from any $t_0 \geqslant s_0$, a run $s_0 \to s_1 \to \cdots \to s_n$ can be simulated "from above" with some $t_0 \to t_1 \to \cdots \to t_n$

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Well-Quasi-Ordering (WQO)

Now what was meant by "$(S, \leqslant)$ is wqo"?

Def. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$.

$\Leftrightarrow$ "*every infinite sequence is a good sequence*"

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Alternatively: $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

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Example. (Dickson's Lemma) $(\mathbb{N}^k,\leqslant_\times)$ is a wqo, with

$$\mathbf{a}=(a_1,\ldots,a_k) \leqslant_\times \mathbf{b}=(b_1,\ldots,b_k) \overset{\text{def}}{\Leftrightarrow} a_1 \leqslant b_1 \wedge \cdots \wedge a_k \leqslant b_k$$

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Example. (Cartesian product) $(X_1 \times \cdots \times X_k, \leqslant_\times)$ is a wqo when $(X_1, \leqslant_1), \ldots, (X_k, \leqslant_k)$ are wqos, with

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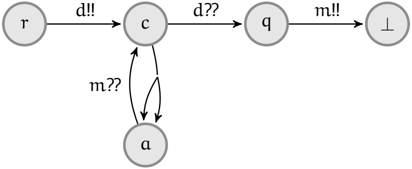
Other important/useful wqos: multisets, trees ordered by embedding (Kruskal's Theorem), and graphs with minors (Robertson & Seymour's Graph Minor Theorem).

Two examples of WSTS

EXAMPLE 1: BROADCAST PROTOCOLS

Broadcast protocols (Esparza et al.'99) are dynamic & distributed collections of finite-state processes communicating via brodcasts and rendez-vous.



A configuration collects the local states of all processes. E.g., $s = \{c, r, c\}$, also denoted $\{c^2, r\}$.

Steps: $\{c^2, q, r\} \to \{a^2, c, q, r\} \to \{a^4, q, r\} \xrightarrow{m} \{c^4, r, \perp\} \xrightarrow{d} \{c, q^4, \perp\}$

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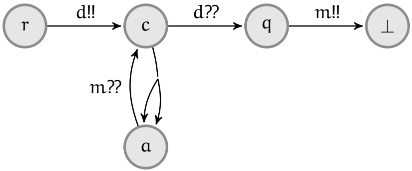

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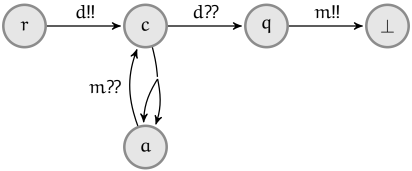

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Brodcast Protocols Are WSTS

Ordering of configurations is multiset inclusion, e.g., $\{c, q\} \subseteq \{c^2, r, q\}$

Fact. *Conf* $= \mathcal{M}_f(\{r, c, a, q, \perp\})$ equipped with $\subseteq$ is a wqo

Proof: this is exactly $(\mathbb{N}^5, \leqslant_\times)$

Fact. Brodcast protocols are monotonic TS

Proof Idea: assume $s_1 \subseteq t_1$ and consider all cases for a step $s_1 \to s_2$

Coro. Broadcast protocols are WSTS

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TPNs have rules like e.g. $\delta = \left\{ \begin{array}{c} c \in [1;2) \\ q \in [2;\infty) \end{array} \mapsto \begin{array}{c} r \in [0;2] \\ q \in [1;1] \\ a \in (0;4) \end{array} \right\}$

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$s' = \{r : 3.0, r : 0.73, q : 1.0, s : 2.1\} \xrightarrow{+0.8} \{r : 3.8, r : 1.53, q : 1.8, a : 2.9\}$

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Fact. Steps are monotonic for multiset inclusion

But $(\mathcal{M}_f(Q \times \mathbb{R}_{\geqslant 0}), \subseteq)$ is not wqo —since already $(\mathbb{R}_{\geqslant 0}, =)$ is not

Timed-arc Nets are WSTS

$$s = \{r : 3.0, r : 0.73, q : 1.0, a : 2.1\} \quad \approx \quad \widetilde{s} = \{r : 3, q : 1\} \bullet \{a : 2\} \bullet \{r : 0\}$$

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$$\begin{array}{ccccccc} [0 & < & x_1 & < & x_2 & < & 1) \\ | & & | & & | & & \\ r:3 & & a:2(+x_1) & & r:0(+x_2) & & \\ q:1 & & & & & & \end{array}$$

In general $\widetilde{s}$ is a sequence over $\mathcal{M}_f(Q \times \{0,1,2,3,4,5+\})$

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Fact. The abstracted system is bisimilar with the original one (NB: durations of time-elapse steps are not preserved).

$$\{r:3, q:1\} \bullet \{a:2\} \bullet \{r:0\} \xrightarrow{+?} \{\} \bullet \{r:3, q:1\} \bullet \{a:2\} \bullet \{r:0\}$$

$$\xrightarrow{+?} \{r:1\} \bullet \{r:3, q:1\} \bullet \{a:2\} \to \cdots$$

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Fact. This new semantics is monotonic wrt pointed sequence embedding $\leqslant_*$ over $\big(\mathcal{M}_f(Q \times \{0, \ldots, 4, 5+\})\big)^+$, a wqo.
Hence TPN are WSTS!!!

Part 2 Verification of WSTS

Termination

Termination is the question, given a TS $\mathcal{S} = (S, \rightarrow, \ldots)$ and a state s_{init}, whether $\mathcal{S}$ has no infinite runs starting from s_{init}

Lem. [Finite Witnesses for Infinite Runs]
A WSTS $\mathcal{S}$ has an infinite run from s_{init} **iff** it has a finite run from s_{init} that is a good sequence

Recall: $s_0, s_1, s_2, \ldots, s_n$ is good $\stackrel{\text{def}}{\Leftrightarrow}$ there exist $i < j$ s.t. $s_i \leqslant s_j$

Proof: "$\Rightarrow$" by def of wqo. "$\Leftarrow$" by simulating $s_i \xrightarrow{+} s_j$ from s_j

$\Rightarrow$ one can decide Termination for a WSTS $\mathcal{S}$ by enumerating all finite runs from s_{init} until a good sequence is found.

NB: This requires some minimal effectiveness assumptions on the WSTS, e.g., that the ordering is decidable

Algorithm extends and allows deciding inevitability, finiteness, and regular simulation

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COVERABILITY

Coverability asks, given $\mathcal{S} = (S, \rightarrow, \ldots)$, a state s_{init} and a target state t, whether $\mathcal{S}$ has a covering run $s_{init} \rightarrow s_1 \rightarrow s_2 \ldots \rightarrow s_n$ with $s_n \geqslant t$.

This is equivalent to having a covering pseudorun of the form

$$s_{init} = s_0 \geqslant t_0 \rightarrow s_1 \geqslant t_1 \rightarrow s_2 \geqslant \cdots t_{n-1} \rightarrow s_n \geqslant t_n = t$$

Fact. In a covering pseudorun, we can assume that each t_i is a minimal (pseudo) predecessor of t_{i+1}

Fact. In a shortest covering pseudorun, the (reversed) sequence $t_n, \ldots, t_1, t_0$ is bad

Lem. [Finite Witnesses for Covering]
A WSTS $\mathcal{S}$ has a covering pseudorun from s_{init} to t **iff** it has one that is minimal and reverse-bad

$\Rightarrow$ one can decide Coverability by enumerating all pseudoruns ending in t (hence backward chaining) that are minimal and reverse-bad.

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Coverability asks, given $\mathcal{S} = (S, \to, \ldots)$, a state s_{init} and a target state t, whether $\mathcal{S}$ has a covering run $s_{init} \to s_1 \to s_2 \ldots \to s_n$ with $s_n \geqslant t$.

This is equivalent to having a covering pseudorun of the form

$$s_{init} = s_0 \geqslant t_0 \to s_1 \geqslant t_1 \to s_2 \geqslant \cdots t_{n-1} \to s_n \geqslant t_n = t$$

Fact. In a covering pseudorun, we can assume that each t_i is a minimal (pseudo) predecessor of t_{i+1}

Fact. In a shortest covering pseudorun, the (reversed) sequence $t_n, \ldots, t_1, t_0$ is bad

Lem. [Finite Witnesses for Covering]
A WSTS $\mathcal{S}$ has a covering pseudorun from s_{init} to t **iff** it has one that is minimal and reverse-bad

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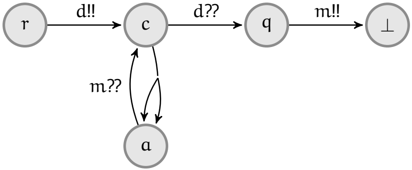
Part 3 Bounding complexity

Broadcast Protocols and Termination



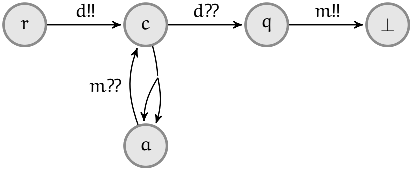
This broadcast protocol terminates: all its runs are bad sequences, hence are finite

Proof. Assume $s_0 \to s_1 \to \cdots \to s_n$ and pick two positions $i < j$.
Write $s_i = \{a^{n_1}, c^{n_2}, q^{n_3}, r^{n_4}, \perp^*\}$, and $s_j = \{a^{n'_1}, c^{n'_2}, q^{n'_3}, r^{n'_4}, \perp^*\}$.

– if $s_i \xrightarrow{+} s_j$ uses only spawn steps then $n'_2 < n_2$,
– if a m and no d have been broadcast, then $n'_3 < n_3$,
– if a d has been broadcast, and then $n'_4 < n_4$.

In all cases, $s_i \not\subseteq s_j$. QED

Broadcast Protocols and Termination



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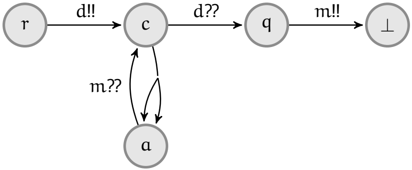

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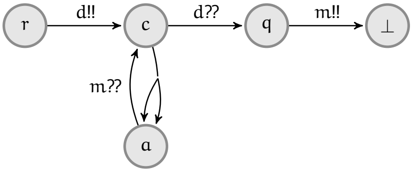
Broadcast Protocols Take Their Time



"Doubling" run: $\{c^{n}, q, (\perp^*)\} \xrightarrow{a^n} \{a^{2n}, q, (\perp^*)\} \xrightarrow{m} \{c^{2n}, (\perp^*)\}$

Building up: $\{c^{2^0}, q^{n}, r\} \xrightarrow{a^{2^0} m} \{c^{2^1}, q^{n-1}, r\} \xrightarrow{a^{2^1} m} \{c^{2^2}, q^{n-2}, r\} \to$
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Then: $\{c, q, r^{n}\} \xrightarrow{*} \{c, q^{2^n}, r^{n-1}\} \xrightarrow{*} \{c, q^{\mathsf{tower}(n)}\}$

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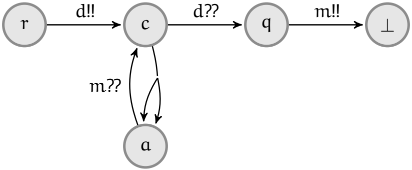

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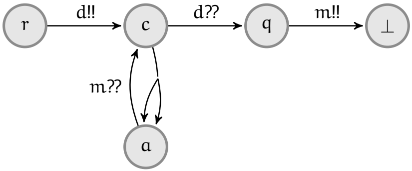

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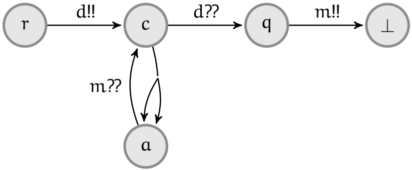

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$\Rightarrow$ Runs of terminating systems may have nonelementary lengths

$\Rightarrow$ Running time of termination verification algorithm is not elementary (for broadcast protocols)

Complexity Analysis?

Key point: When analyzing the termination algorithm, the main question is "how long can a bad sequence be?"

WQO-theory only says that a bad sequence is finite

Over $(\mathbb{N}^k, \leqslant_\times)$, one can find arbitrarily long bad sequences:

— 999, 998, . . . , 1, 0

— $(2,2)$, $(2,1)$, $(2,0)$, $(1,999)$, . . . , $(1,0)$, $(0,999999999)$, . . .

Two tricks: unbounded start element, or unbounded increase in a step

The runs of a broadcast protocol don't play these tricks!

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CONTROLLED BAD SEQUENCES

Def. A control is a pair of $n_0 \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$.

Def. A sequence $x_0, x_1, \ldots$ is controlled $\overset{\text{def}}{\Leftrightarrow}$ $|x_i| \leqslant g^i(n_0)$ for all $i = 0, 1, \ldots$

Fact. For a fixed wqo $(A, \leqslant, |.|)$ and control (n_0, g), there is a bound on the length of controlled bad sequences.

Write $L_{g,A}(n_0)$ for this maximum length.

Length Function Theorem for $(\mathbb{N}^k, \leqslant_\times)$:

— $L_{g,\mathbb{N}^k}(n_0) \leqslant g^{\omega^k}(n_0)$

— $L_{g,\mathbb{N}^k}$ is in $\mathscr{F}_{k+m-1}$ for g in $\mathscr{F}_m$ [Figueira^{2}SS'11]

(more later on Fast-Growing Hierarchy)

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APPLYING TO BROADCAST PROTOCOLS

Fact. The runs explored by the Termination algorithm are controlled with $n_0 = |s_{init}|$ and $g = \mathit{Succ} : \mathbb{N} \to \mathbb{N}$.

$\Rightarrow$ Time/space bound in $\mathscr{F}_{k-1}$ for broadcast protocols with k states, and in $\mathscr{F}_\omega$ when k is not fixed.

Fact. The minimal pseudoruns explored by the backward-chaining Coverability algorithm are controlled by $|t|$ and *Succ*.

$\Rightarrow \cdots$ *same upper bounds* $\cdots$

This is a general situation:
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NOW APPLYING TO TIMED-ARC NETS

Fact. The runs of a Timed-arc net N are controlled with $n_0 = |s_{init}|$ and $g : x \mapsto x + |N|$,
or with $n_0 = |s_{init}| + |N|$ and $g = \mathit{Double} : x \mapsto 2x$ if we want fixed g.

For *Conf* $= \mathcal{M}_f(Q \times \{0, 1, \ldots, M+\})^*$ ordered with pointed sequence embedding, the Length Function theorem [SS '11] gives

$$L_{g,\mathit{Conf}} \text{ in } \mathscr{F}_{\omega^{\omega^k}} \text{ where } k = |Q| \times M$$

$\Rightarrow$ Time/space bound in $\mathscr{F}_{\omega^{\omega^\omega}}$ for Timed-arc Nets verification

These bounds are optimal!
— Verification of Timed-arc nets is $\mathscr{F}_{\omega^{\omega^\omega}}$-complete [HSS '12]
— Verification of Broadcast protocols is $\mathscr{F}_{\omega}$-complete, or "Ackermann-complete" [S '10]

Bottom line: we can provide definite complexity for many WSTS models

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THE FAST-GROWING HIERARCHY

An ordinal-indexed family $(F_\alpha)_{\alpha \in \mathit{Ord}}$ of functions $\mathbb{N} \to \mathbb{N}$

$$F_0(x) \stackrel{\text{def}}{=} x + 1 \qquad F_{\alpha+1}(x) \stackrel{\text{def}}{=} \overbrace{F_\alpha(F_\alpha(\ldots F_\alpha}^{x+1}(x)\ldots))$$

$$F_\omega(x) \stackrel{\text{def}}{=} F_{x+1}(x)$$

gives $F_1(x) \sim 2x$, $F_2(x) \sim 2^x$, $F_3(x) \sim \text{tower}(x)$ and $F_\omega(x) \sim \text{ACKERMANN}(x)$, the first F_α that is not primitive recursive.

$F_\lambda(x) \stackrel{\text{def}}{=} F_{\lambda_x}(x)$ for λ a limit ordinal with a fundamental sequence $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda$.

E.g. $F_{\omega^2}(x) = F_{\omega\cdot(x+1)}(x) = F_{\omega\cdot x+x+1}(x) = \overbrace{F_{\omega\cdot x+x}(F_{\omega\cdot x+x}(..F_{\omega\cdot x+x}}^{x+1}(x)..))$

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CONCLUDING REMARKS

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- WSTS allow complexity analysis

Join the fun!

Technical details are lighter than it seems.
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THANK YOU FOR YOUR INTEREST

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