Symposium for Gordon Plotkin

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From Partial Lambda-calculus to Monads

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AIM: recall influence of Plotkin (and others) on my PhD research (and beyond)

- Mainly overview of published work + personal comments and opinions

  Please interrupt to correct my account or to add your comments

  Main focus on

- Partial lambda-calculus [Mog88] 1984-1988

  but work placed in broader context:

  - partiality in: Logic, Algebra, Computability
  - lambda-calculi as: PL [Lan66, Plo75], ML [Sco93, GMW79, Plo85]
  - domain theory [FJM^{+}96]: classical, axiomatic, synthetic

- Applications of monads [Mac71, Man76]

  - for computational types (lifting and recursion) [Mog89, Mog91] 1988-...
  - in pure functional languages (Haskell) – Wadler et al.
  - for collection types (in databases) – Buneman et al.

  including recent contributions by Plotkin et al. [HPP02]

Partial Lambda-Calculus: Background

Relevant work (1983-1985)

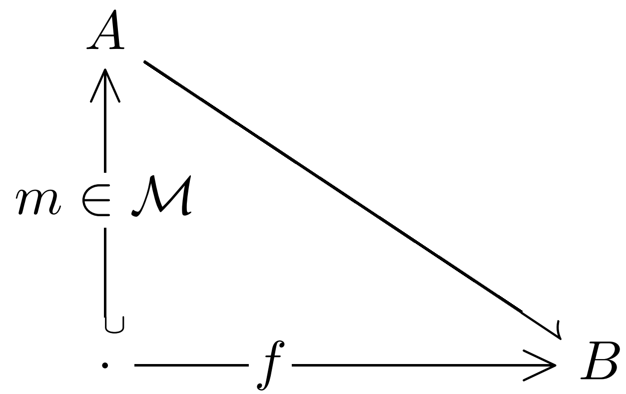
- 1983 categories of partial maps for computability [dPH86, LM84]

  1985 the cleanest exposition on categories of partial maps [RR88]

  $\mathcal{C}$ category (of total maps) $f\colon A \longrightarrow B$

  $\mathcal{M}$ dominion = class of monos (with certain properties)

  $\mathcal{C}_p$ category $\mathcal{P}(\mathcal{C},\mathcal{M})$ of partial maps



  $\top\colon\ 1 \longrightarrow \Sigma$ dominance classifying $\mathcal{M}$ (in topos $\Sigma \subset \Omega$ with certain properties)
  central role in Synthetic Domain Theory [Hyl91, Pho90, Pho91]

Partial Lambda-Calculus: Background

Relevant work (1983-1985)

- 1983 categories of partial maps
- 1984 reformulation of domain theory using partial continuous maps [Plo85]
  Hand-written notes for the TPG course
  - computational lambda-calculus [Mog89] influenced by metalanguage
    $\tau ::= \ldots \mid \tau_\perp \mid \tau_1 \rightharpoonup \tau_2 \quad e ::= \ldots \mid [e] \mid$ let $[x]$ be e in e'
    and its interpretation in $\mathbf{Cpo}_p$
  - Axiomatic Domain Theory [Fio94] influenced by reformulation:
    $\mathbf{Cpo}_p$ paradigmatic example of algebraically compact category [Fre92]

Partial Lambda-Calculus: Background

Relevant work (1983-1985)

- 1983 categories of partial maps
- 1984 reformulation of domain theory using partial continuous maps [Plo85]

Previous relevant work (. . . -1983)

- 1982 more systematic study of categories of partial maps [Obt86, CO87]
- partiality in algebraic specifications: [Bur82]
- partiality in (intuitionistic) logic: LPE [Fou77, Sco79], LPT [Bee85]
- mismatch between lambda-calculus and programming languages [Plo75]

λ_V CBV axioms $(\lambda x.t)v \longrightarrow t[x{:=}v]$ and $(\lambda x.vx) \longrightarrow v$ with $v ::= x \mid \lambda x.t$ values

- λ_p is derived from models like λ_V is derived from operational semantics
- $\lambda_V \subset \lambda_c \subset \lambda_p$ are correct (but incomplete) for CBV (on untyped λ-terms)
- λ_c on (simply typed) λ-terms is inverse image of $\lambda\beta\eta$ w.r.t. CBV CPS [SF93]

Partial Lambda-Calculus: Approach and Results

- Systematic and *unbiased* investigation of partiality
  - in the setting of both intuitionistic and classical logic
  - in set-theoretic models and order-theoretic models ($\mathbf{PoSet}$ or $\mathbf{Cpo}$)

Axiomatization in Logic of Partial Terms (LPT) [Bee85]

  - variables x range on *existing elements*, but terms e could be undefined (in LPE free x range on *partial elements*, and e denote partial elements)
  - $e\downarrow$ means "e defined" (or "evaluation of e terminates")
    $e_1 = e_2$ means "e_1 and e_2 defined and equal" (similarly $e_1 \leq e_2$)
    $e_1 \simeq e_2 \overset{\Delta}{\iff} (e_1\downarrow \vee e_2\downarrow) \supset e_1 = e_2$ (but $e_1 \lesssim e_2 \overset{\Delta}{\iff} e_1\downarrow \supset e_1 \leq e_2$)

Partial Lambda-Calculus: Approach and Results

- Systematic and *unbiased* investigation of partiality
  $\lambda_p\beta\eta$-models as partial Combinatory Algebras (pCA) + extensionality
  For $\lambda_p\beta$-models adapt FOL axiomatization of $\lambda\beta$-models in [Mey82]

pCA $Kxy = x \quad (Sxy)\downarrow \quad Sxyz \simeq xz(yz)$
abstraction $[x]e$ s.t. $([x]e)\downarrow$ and $([x]e)x \simeq e$ definable by induction [Bar84]

ext.$\simeq$ $(\forall z.xz \simeq yz) \supset x = y$

ext.$\lesssim$ $(\forall z.xz \lesssim yz) \supset x \leq y$ (and monotonicity $x_1 \leq x_2 \wedge y_1 \leq y_2 \supset x_1y_1 \lesssim x_2y_2$)

tot $(xy)\downarrow$ (i.e. application always defined) – $\lambda\beta\eta$ = $\lambda_p\beta\eta$ + tot ($\perp$ not built-in)

	intuitionistic	classical
set-theoretic	$J\lambda_p\beta\eta$	$K\lambda_p\beta\eta$
order-theoretic	$J\lambda_p^{\leq}\beta\eta$	$K\lambda_p^{\leq}\beta\eta$

Partial Lambda-Calculus: Approach and Results

- Systematic and *unbiased* investigation of partiality
- Comparison among λ_p-calculi (and also λ- and λ_V-calculi)
  - on equations $e_1 = e_2$ between pure λ-terms – quite complex

$$\begin{array}{ccccccc} \lambda_V\beta\eta & \subset & J\lambda_p\beta\eta & \subset & J\lambda_p^{\leq}\beta\eta & & \\ & & \cap & & \cap & & \\ & & K\lambda_p\beta\eta & \subset & K\lambda_p^{\leq}\beta\eta & \subset & \lambda\beta\eta(= J\lambda\beta\eta = K\lambda^{\leq}\beta\eta) \end{array}$$

  - on definedness assertions $e \downarrow$ for pure λ-terms

$$\lambda_V\beta\eta \quad = \quad J\lambda_p\beta\eta \quad = \quad K\lambda_p^{\leq}\beta\eta \quad \subset \quad \lambda\beta\eta$$

Corollary of computational adequacy result in [Plo85]

Partial Lambda-Calculus: Approach and Results

- Systematic and *unbiased* investigation of partiality
- Analogies between λ_p-calculi and λ-calculus (results and proof techniques)

  Completeness results for type hierarchies (simply typed fragment)
  - $\lambda\beta\eta$ complete for $\mathbf{Set}(N)$ [Fri75] and $\mathbf{Cpo}(N_\perp)$ [Plo82]
  - $K\lambda_p\beta\eta$ complete for $\mathbf{Set}_p(N)$ – partial functions
  - $K\lambda_p^{\leq}\beta\eta$ complete (on inequalities) for $\mathbf{PoSet}_p(P(N))$

    Unknown if similar result holds in $\mathbf{Cpo}_p$

  Results proved using *logical relations* for λ_p-calculus

  $fR_{\tau\rightharpoonup\tau'}g \stackrel{\Delta}{\Longleftrightarrow} \forall x,y.\, xR_\tau y \supset fx\widetilde{R}_{\tau'}gy$ where $e_1\widetilde{R}e_2 \stackrel{\Delta}{\Longleftrightarrow} (e_1\downarrow \vee e_2\downarrow) \supset e_1Re_2$

Partial Lambda-Calculus: Approach and Results

- Systematic and *unbiased* investigation of partiality
- Analogies between λ_p-calculi and λ-calculus (results and proof techniques)

  Characterization of equality (on untyped pλ-terms) by confluent reduction

  Very complex in comparison with characterization of equality in $\lambda\beta\eta$

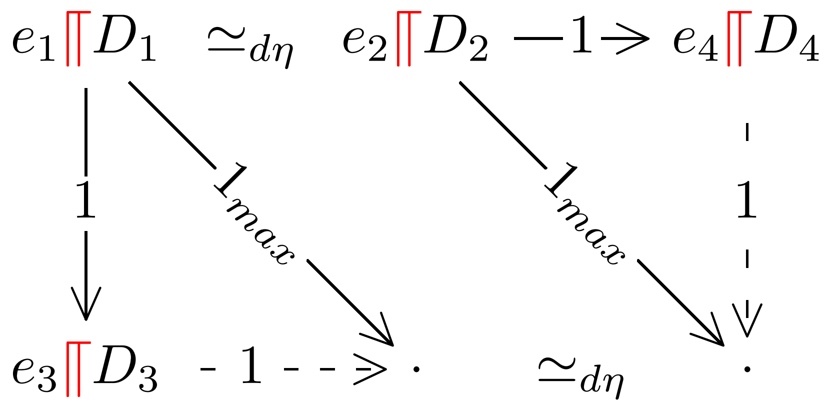
$$e \| D \qquad \frac{e_1 \| D_1 \qquad e_2 \| D_2}{e_1 e_2 \| \underline{\{e_1 e_2\} \cup D_1 \cup D_2}} \qquad \frac{e \| \underline{D_1 \cup D_2}}{(\lambda x.e \| D_1) \| D_2}\; x \notin \mathrm{FV}(D_2) \quad \ldots \text{ [Obt86]}$$

$$\beta_p \quad (\lambda x.e_1 \| D_1) e_2 \| \underline{\{(\lambda x.e_1 \| D_1) e_2\} \cup D_2} \longrightarrow e_1[x{:}{=}e_2] \| \underline{(D_1[x{:}{=}e_2]) \cup D_2}$$

On pλ-terms one should use one-step parallel β_p-reduction $\;\xrightarrow{}_1\!\!>$

- $\simeq$ in $J\lambda_p\beta\eta$ characterized by $\;-_1\!\!\!>\;$ and decidable equivalence $\simeq_{d\eta}$ [Per88]
- $\lesssim$ in $J\lambda_p^{\leq}\beta\eta$ characterized by $\;-_1\!\!\!>\;$ and decidable preorder $\lesssim_{d\eta}$

diamond property of $\;-_1\!\!\!>$
up to $\simeq_{d\eta}$ (or $\lesssim_{d\eta}$)



Partial Lambda-Calculus: Concluding Remarks

- Too much focus on partiality!
  - λ_p-calculus is not sound for reasoning on CBV languages (e.g. SML) with other computational effects, while λ_V-calculus is still sound

  Generalize $\Longrightarrow$ Notions of computations as Monads

  Work in this direction started after PhD submission, and was discussed during the PhD exam (Hyland, Milner, Plotkin). Essential contributions in early stages:
  - Plotkin $\Longrightarrow$ lots of examples inspired by Denotational Semantics
    Essential to get reassurance that we were on the right track
  - Hyland, Kock,... $\Longrightarrow$ pointers to the relevant Category Theory literature
    Basically all the necessary mathematics was already there [Koc72, Man76]

Partial Lambda-Calculus: Concluding Remarks

- Too much focus on partiality!
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  Generalize $\Longrightarrow$ Notions of computations as Monads

- Too generic on what kind of partiality!
  - λ_p-calculus does not adequately capture partiality in the setting of computability and domain theory

  Specialize $\Longrightarrow$ Axiomatic and Synthetic Domain Theory

ADT Axioms for a dominance Σ in a (order-enriched) category [FP94, Fio94] to interpret a metalanguage (e.g. FPC) with recursive definitions and recursive types (in the category of partial maps [Fre90, Fre92])

SDT Axioms for a dominance $\Sigma \subset \Omega$ in a topos to ensure existence of full sub-category of domains (Scott's slogan "domains as sets") [Hyl91, Pho90, Pho91] (some axioms incompatible with classical logic)

Monads and Computational Types

- $\mathcal{C}$ category of *sets and maps*, τ set of *values* and $M\tau$ set of *programs* of type τ
- M should have (at least) the structure of a monad

$$\frac{e\colon\tau}{\textit{ret}\ e\colon M\tau} \qquad \frac{e_1\colon M\tau_1 \quad x\colon\tau_1 \vdash e_2\colon M\tau_2}{\textit{do}\ x \leftarrow e_1;\ e_2\colon M\tau_2}$$

$$\begin{aligned}
\textit{do}\ x \leftarrow (\textit{ret}\ e_1);\ e_2 &= e_2[x\colon=e_1] \\
\textit{do}\ x_2 \leftarrow (\textit{do}\ x_1 \leftarrow e_1;\ e_2);\ e_3 &= \textit{do}\ x_1 \leftarrow e_1;\ (\textit{do}\ x_2 \leftarrow e_2;\ e_3) \quad x_1 \notin \mathrm{FV}(e_3) \\
\textit{do}\ x \leftarrow e;\ (\textit{ret}\ x) &= e
\end{aligned}$$

PL CBV programming language interpreted in Kleisli category $\mathcal{C}_M$ for a given M

Like interpretation of Plotkin's metalanguage in $\mathbf{Cpo}_p$ [Plo85]

ML_M Monadic metalanguage interpreted in $\mathcal{C}$ (possibly with several monads)

extends *conservatively* a typed λ-calculus for $\mathcal{C}$ with computational types

Semantics of programming languages via translation in ML_M followed by interpretation of M

Monads and Computational Types

- $\mathcal{C}$ category of *sets and maps*, τ set of *values* and $M\tau$ set of *programs* of type τ
- M should have (at least) the structure of a monad
- Relation to partiality: partial map classifier $\mathcal{C}(A, B_\perp) \cong \mathcal{C}_p(A, B)$ is a monad
  the category $\mathcal{C}_p$ of partial maps isomorphic to the Kleisli category $\mathcal{C}_\perp$
  monad morphism $(-)_\perp \stackrel{\cdot}{\rightarrow} M$ in model for Axiomatic/Synthetic Domain Theory
  $\Longrightarrow$ recursive definitions of elements in $M\tau$ (is a EM-algebra for $(-)_\perp$)
  M usable in recursive domain equations (as M extends to $\mathcal{C}_\perp$)
- Relation to λ_V: CBV translation $(-)^v$ in ML_M with $V = (MV)^V$ – $x_i\colon V \vdash e^v\colon MV$
  - $\lambda_V \subset \lambda_c =$ calculus on (untyped) λ-terms induced by $(-)^v$

[SF93] $\lambda_c =$ inverse image of $\lambda\beta\eta$ w.r.t. CBV CPS translation $\overline{(-)}$ of [Plo75]
Continuations as worst case instance of computational types
Corollary: completeness results for simply typed fragment
ML_M complete for $\mathbf{Set}$ and $\mathbf{Cpo}$ with monad $M- = A^{(A^-)}$
In $\mathbf{Set}$ interpret A and base type with N, in $\mathbf{Cpo}$ with $N_\perp$

Monads in Functional Programming

- *Extension* of Haskell with computational types [Wad90, Wad92a, Wad92b]

  Motivation: to *mimic* impure features in a pure functional language
  and also hide how much impure features to mimic

  Analogy with monadic metalanguage extending *conservatively* λ-calculus

- Further ideas originally developed within Haskell
  - Monadic encapsulation of effects *run*: $(\forall\alpha.M_\alpha\tau) \to \tau$ [LP94, LS97, MS01]
    No need to change Haskell's type system with a type-and-effect system [LG88, TJ94]
  - *Refinement* of computational types with effects $M_\epsilon\tau$ [Wad98, BHM02]
    Refined type *semantically* $(A \in \mathbf{Set}, P \subseteq A)$, *syntactically* $|M_\epsilon\tau| = M|\tau|$
  - Monadic *value recursion* $\dfrac{x\colon M\tau \vdash e\colon M\tau}{\textit{Mfix}\ x.e\colon M\tau}$ [EL00, Erk02, MS04]

    Operational semantics of *Mfix* , only *loose axiomatization*
    Mfix $x.$*ret* $e =$ *fix* $x.$*ret* e *Mfix* $(x_2.$*do* $x_1 \leftarrow c; e) =$ *do* $x_1 \leftarrow c;$ (*Mfix* $x_2.e)$

Monads and Collection Types [BNTW95]

- From list comprehension to monad comprehension [Wad92a]

$$\frac{x_1{:}M\tau_1,\ldots,x_{j-1}{:}M\tau_{j-1}\vdash e_j{:}M\tau_j \quad 1\leq j\leq n \qquad x_1{:}M\tau_1,\ldots,x_n{:}M\tau_n\vdash e{:}\tau}{\{e|x_1\leftarrow e_1,\ldots,x_n\leftarrow e_n\}{:}M\tau \quad \text{comprehension notation} \qquad \textit{do}\ x_1\leftarrow e_1;\ldots;x_n\leftarrow e_n;\ \textit{ret}\ e{:}M\tau \quad \text{do-notation}}$$

  - $c\in M\tau$ collection of elements in τ – $M\tau$ collection type
  - $c\in M\tau$ computes values in τ – $M\tau$ computational type
- A collection M should have (at least) the structure of a monad, but also
  - a collection $c\in M\tau$ should have a finite numbers of elements
  - empty collection $0{:}M\tau$ and union $c_1+c_2{:}M\tau$ of two collections $c_1,c_2{:}M\tau$

  [Man98] defines and studies collection monads in $\mathbf{Set}$

  M collection monad $\Longleftrightarrow$ induced by a *balanced* algebraic theory (Σ,E)
  i.e. $\mathrm{FV}(e_1)=\mathrm{FV}(e_2)$ for all equations $(e_1,e_2)\in E$
- M collection monad $\Longrightarrow mem_X{:}MX\longrightarrow\mathcal{P}_{fin}(X)$ monad morphism

Combining Monads

- Motivation: modular approach to semantics of programming languages

  ideally semantics as easy to extend as syntax of PL [Mos90]
- Approach: build monad for a complex language from simple monads
  Ideally from monads capturing *one* computational effect

  Early proposals, e.g. monad transformers [BHM02]:
  methodologically and mathematically unsatisfactory

Combining Monads

- Motivation: modular approach to semantics of programming languages
- New proposal [PP02, PP03, HPP02]:
  revisits correspondence between *algebraic theories* and monads
  Originally the correspondence was used to establish computational adequacy between denotational and operational semantics for a monadic language with *algebraic* effects.
- May identify better monads, e.g. monad MX induced by read/write operations $lkp\colon L \longrightarrow MV$ and $upd\colon L, V \longrightarrow M1$ on set L of global variables is strictly included in naive state monad, when L is infinite
  - $MX \subset (X \times S)^S$ with $S = V^L$
    $c \in MX \iff \forall s_1.\exists R, W \subseteq_{fin} L.\forall s_2.s_1 =_R s_2 \supset cs_2 = (x_1, s'_2)$
    where $(x_1, s'_1) = cs_1$ and s'_2 s.t. $s'_2 =_W s'_1$ and $s'_2 =_{\overline{W}} s_2$
- Main limitation: continuation monads don't fit in algebraic framework
- Technical complications: need to go beyond plain algebraic theories
  Enriched setting (e.g. **Cpo**-enrichment), and arities not necessarily finite (e.g. countable).

Combining Monads

- Motivation: modular approach to semantics of programming languages
- New proposal [PP02, PP03, HPP02]:
  revisits correspondence between *algebraic theories* and monads
- Mathematically clean: use few natural combinations for algebraic theories
  - $+$ disjoint union of operations and equations of two theories T_1 and T_2
  - $\otimes$ disjoint union of operations and equations + equations for commutativity

$$op_1(op_2(x_{i,j}|j \in n)|i \in m) = op_2(op_1(x_{i,j}|i \in m)|j \in n)$$

  operation op_1 of theory T_1 commutes with op_2 of theory T_2

  Most monad transformers defined using (conterpart of) $+$ and $\otimes$ (on monads).

Combining Monads

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  operation op_1 of theory T_1 commutes with op_2 of theory T_2
- The approach can be used for a modular approach to collection types too!
  in this case plain algebraic theories (finitary monads on $\mathbf{Set}$) suffice [Man98]
  $+$ and $\otimes$ preserve balanced algebraic theories
  problematic monads (like continuations) are ruled out by application domain.

Conclusions

- I would like to conclude by mentioning a very influential *unpublished* work by

  Rod Burstall
  Robin Milner
  Gordon Plotkin et al.
- well-known worldwide, and
- particularly appreciated by CS researchers that had the opportunity to be or visit Edinburgh in the last 20 years

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The Laboratory for Foundations of Computer Science

Many Thanks, Gordon!

Technical Details: CBV

- λ_V-calculus terms $t{::=}\,x \mid \lambda x.t \mid t_1 t_2$ values $v{::=}\,x \mid \lambda x.t$ and axioms

$$\beta_V \;\; (\lambda x.t)v = t[x{:=}v] \qquad \eta_V \;\; (\lambda x.vx) = v \text{ if } x \notin \mathrm{FV}(v)$$

- CBV translation $(-)^v$ in ML_M and CBV CPS translation $\overline{(-)}$ in λ-calculus

t	x	$\lambda x.t$	$t_1 t_2$
t^v	*ret* x	*ret* $(\lambda x{:}V.t^v)$	*do* $x_1 \leftarrow t_1^v;\ x_2 \leftarrow t_2^v; x_1 x_2$
$\overline{t}$	$\lambda k.kx$	$\lambda k.k(\lambda x.\overline{t})$	$\lambda k.\overline{t}_1(\lambda x_1.\overline{t}_2(\lambda x_2.x_1 x_2 k))$

Monad of continuations $M\tau = A^{(A^\tau)}$ with answers in A

$$\textit{ret}\ e = \lambda k.ke \qquad \textit{do}\ x \leftarrow e_1; e_2 = \lambda k.e_1(\lambda x.e_2 k)$$

Technical Details: CBV

- λ_V-calculus terms $t ::= x \mid \lambda x.t \mid t_1 t_2$ values $v ::= x \mid \lambda x.t$ and axioms

$$\beta_V \quad (\lambda x.t)v = t[x := v] \qquad \eta_V \quad (\lambda x.vx) = v \text{ if } x \notin \mathrm{FV}(v)$$

- inverse image of $\lambda\beta\eta$ w.r.t. CBV CPS translation [SF93] $\beta_V + \eta_V +$

$$\begin{array}{rcll} (\lambda x.x)t &=& t & \\ E[(\lambda x.t_1)t_2] &=& (\lambda x.E[t_1])t_2 & x \notin \mathrm{FV}(E) \\ E[t_1 t_2 t_3] &=& (\lambda x.E[x t_3])(t_1 t_2) & x \notin \mathrm{FV}(E, t_3) \\ (\lambda x.E[x' x])t &=& E[x' t] & x \notin \mathrm{FV}(E, x') \end{array}$$

where $E ::= [] \mid vE \mid Et$ CBV evaluation contexts, or alternatively

$$\begin{array}{rcll} (\lambda x.x)t &=& t & \\ (\lambda x_2.t_3)((\lambda x_1.t_2)t_1) &=& (\lambda x_1.(\lambda x_2.t_3)t_2)t_1 & x_1 \notin \mathrm{FV}(t_3) \\ t_1 t_2 t_3 &=& (\lambda x.x t_3)(t_1 t_2) & x \notin \mathrm{FV}(t_3) \\ v t_1 t_2 &=& (\lambda x.vx)(t_1 t_2) & x \notin \mathrm{FV}(v) \end{array}$$

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