System Analysis and Robustness

Eugenio Moggi[1⋆], Amin Farjudian[2], and Walid Taha[3]

[1] DIBRIS, Genova Univ., Genova, Italy, moggi@unige.it
[2] Univ. of Nottingham Ningbo China, amin.farjudian@nottingham.edu.cn
[3] Halmstad Univ., Halmstad, Sweden, walid.taha@hh.se

Abstract. Software is increasingly embedded in a variety of physical contexts. This imposes new requirements on tools that support the design and analysis of systems. For instance, modeling embedded and cyber-physical systems needs to blend discrete mathematics, which is suitable for modeling digital components, with continuous mathematics, used for modeling physical components. This blending of continuous and discrete creates challenges that are absent when the discrete or the continuous setting are considered in isolation. We consider robustness, that is, the ability of an analysis of a model to cope with small amounts of imprecision in the model. Formally, we identify analyses with monotonic maps between complete lattices (a mathematical framework used for abstract interpretation and static analysis) and define robustness for monotonic maps between complete lattices of closed subsets of a metric space.

Keywords: Analyses; Robustness; Domain theory.

1 Introduction

In a discrete setting one can achieve absolute precision[4], in a continuous setting there are two pervasive and unavoidable sources of imprecision:

1. imprecision in measurements, namely predictions based on a mathematical model and observations on a *real system* can be compared only up to the precision of instruments used for measurements on the real system, and
2. imprecision in representing continuous quantities in computer-assisted tools for modeling and analyzing hybrid/continuous systems.

Thus, a real number $x{:}\,\mathbb{R}$ in mathematics, becomes $x \pm \epsilon$ in physics, with $\epsilon > 0$ *measurement error*, in theory of computation becomes an interval $[\underline{x}, \overline{x}]$ with $\underline{x}$ and $\overline{x}$ belonging to a subset of $\mathbb{R}$ with exact finite representations (e.g., floating-point or rational numbers) [8][5]. However, any $x{:}\,\mathbb{R}$ can be **approximated** by

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[4] This does not exclude the possibility of using *imprecise* (aka loose) specifications.

[5] Representing a real with a float, as done in traditional numerical methods, means that the imprecision in computations is either ignored or is tracked manually.

proper rational intervals $[\underline{x}, \overline{x}]$ with **arbitrarily small imprecision**, i.e., for any $\delta > 0$ there are rational numbers $\underline{x}$ and $\overline{x}$ such that $\underline{x} < x < \overline{x}$ and $0 < \overline{x} - \underline{x} < \delta$.

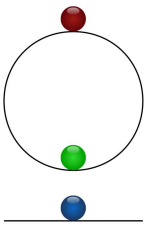
Approximability extends to continuous maps on $\mathbb{R}$. First, a continuous map f on $\mathbb{R}$ has a Scott continuous *natural extension* $\overline{f}(I) \triangleq \{f(x) | x : I\}$ on the cpo $\mathbb{IR}$ of intervals ordered by reverse inclusion. Scott continuity implies that the imprecision of $\overline{f}(I)$ goes to 0 when the imprecision of I goes to 0. Second, $\overline{f}$ can be replaced by a Scott continuous F mapping proper rational intervals to proper rational intervals such that $F([x]) = [f(x)] = \overline{f}([x])$, thus $\overline{f}(I) \subseteq F(I)$. When f is not continuous, one can find a monotonic F on $\mathbb{IR}$ such that:

1. $\forall x : \mathbb{R}. F([x]) = [f(x)]$, but F fails to be Scott continuous, or
2. F is Scott continuous, $\forall I : \mathbb{IR}. \overline{f}(I) \subseteq F(I)$, but $\forall x : \mathbb{R}. F([x]) = [f(x)]$ fails.

In both cases "$F(I)$ converges to $f(x)$ when I converges to x" fails.

Robustness. In [6], we introduced **robustness**, a property of monotonic maps between complete lattices of (closed) subsets in metric spaces. Intuitively, robustness requires that *small changes* to the input I of a map F cause small changes to its output, where the definition of small relies on the metrics. Often, analyses can be identified with monotonic maps between complete lattices. For instance, reachability analysis can be cast as a monotonic map F on the complete lattice $\mathbb{P}(\mathbb{S})$ of subsets of the state space $\mathbb{S}$, that takes a set I of initial states and outputs the set $R(I)$ of states reachable from I, thus $I \subseteq R(I) = R^2(I)$.

If $\mathbb{S}$ is a metric space, then one has the mathematical framework to measure imprecision. The picture below shows the initial state s of three systems (red, green and blue) consisting of a ball that can move (in a one-dimensional space) under the effect of gravity. We assume that initially the speed is 0, thus from s only s is reachable, i.e., $R_r(\{s\}) = R_g(\{s\}) = R_b(\{s\}) = \{s\}$, but:



- the red ball (top) is *unstable*, i.e., a small change s' to s means that $R_r(\{s'\})$ includes some states far from s;
- the green ball (middle) is *stable*, i.e., a small change s' to s implies that all states in $R_g(\{s'\})$ are close to s;
- the blue ball (bottom) is stable, if a small change s' affects only the position (while the speed remains 0); it is unstable, if the speed can change (and there is no friction).

These claims on s can be recast as follows: R_g is *robust at* $\{s\}$, R_r is not.

Contributions. This paper presents mainly results published in [6,7], namely:

1. A definition of imprecision in the context of metric spaces (Sec 2), related to the *noise model* in [3] and δ-safety in [5]. The main point is that imprecision makes a subset S of a metric space $\mathbb{S}$ indistinguishable from its closure $\overline{S}$.
2. A notion of robustness [6] (Sec 3) for monotonic maps $A : \mathbb{C}(\mathbb{S}_1) \to \mathbb{C}(\mathbb{S}_2)$, the restriction to closed subsets is due to indistinguishability of S and $\overline{S}$.

Moreover, it includes a result (Thm 1 in Sec 4), which subsumes those in [6,7] and provides an *almost* complete picture on existence of *best* robust approximations.

2 Imprecision in Metric Spaces

Definition 1. *Given a metric space $\mathbb{S}$, with distance function d, we define:*

1. $B(S,\delta) \triangleq \{y | \exists x{:}\, S.d(x,y) < \delta\}$, *where* $S{:}\,\mathsf{P}(\mathbb{S})$ *and* $\delta > 0$. *Intuitively,* $B(S,\delta)$ *is the set of points in* S *with imprecision* $< \delta$. $B(S,\delta)$ *is open, because it is the union of open balls* $B(s,\delta)$ *with* $s{:}\,S$, *moreover* $B(B(S,\delta),\delta') \subseteq B(S,\delta+\delta')$.
2. $\overline{S}{:}\,\mathsf{C}(\mathbb{S})$ *is the* ***closure*** *of* $S{:}\,\mathsf{P}(\mathbb{S})$, *i.e., the smallest* $C{:}\,\mathsf{C}(\mathbb{S})$ *such that* $S \subseteq C$. *For* $S{:}\,\mathsf{P}(\mathbb{S})$ *and* $\delta > 0$ *the following holds:* $S \subseteq \overline{S} \subseteq B(S,\delta) = B(\overline{S},\delta)$. *Thus, in the presence of imprecision,* S *and* $\overline{S}$ *are* ***indistinguishable****.*
3. $S_\delta \triangleq \overline{B(S,\delta)}$ *is the* δ***-fattening*** *of* $S{:}\,\mathsf{P}(\mathbb{S})$. *Intuitively,* S_δ *is the set of points in* S *with imprecision* $\leq \delta$. *In fact,* $B(S,\delta) \subseteq S_\delta \subseteq B(S,\delta')$ *when* $0 < \delta < \delta'$. *For* $S{:}\,\mathsf{P}(\mathbb{S})$ *the following holds:* $\overline{S} = \bigcap_{\delta>0} B(S,\delta) = \bigcap_{\delta>0} S_\delta$. *Thus, the closure* $\overline{S}$ *is the set of points that are in* S *with arbitrarily small imprecision.*

We consider some examples of metric spaces motivated by applications.

Example 1 (Discrete). A set $\mathbb{S}$ can be viewed as a *discrete* metric space, i.e., $d(s,s') = 1$ when $s \neq s'$, and any subset is closed and open. Thus, $\mathsf{C}(\mathbb{S}) = \mathsf{P}(\mathbb{S})$. More generally, if $\mathbb{S}$ is δ**-discrete**, i.e., $\forall s,s'{:}\,\mathbb{S}.s \neq s' \implies \delta \leq d(s,s')$, then $\forall S{:}\,\mathsf{P}(\mathbb{S}).S_\delta = S$, i.e., an imprecision $\leq \delta$ amounts to absolute precision.

Example 2 (Euclidean). Euclidean spaces $\mathbb{R}^n$ (and Banach spaces) are used for modeling continuous and hybrid systems [4]. For $C{:}\,\mathsf{C}(\mathbb{R}^n)$, δ-fattening has a simpler alternative definition, namely $C_\delta = \{y | \exists x{:}\, C.d(x,y) \leq \delta\}$.

Example 3 (Products, sub-spaces, sums). The product $\mathbb{S}_0 \times \mathbb{S}_1$ of two metric spaces is the product of the underlying sets with metric $d(x,y) \triangleq \max_{i:2} d_i(x_i,y_i)$.

A subset S' of $\mathbb{S}$ inherits the metric, thus can be considered a metric space $\mathbb{S}'$. If S' is also closed, then $\mathsf{C}(\mathbb{S}') \subseteq \mathsf{C}(\mathbb{S})$ and the δ-fattening of $S{:}\,\mathsf{P}(\mathbb{S}')$ is $S_\delta \cap S'$.

The sum $\coprod_{i:I} \mathbb{S}_i$ of an I-indexed family of metric spaces is $\{(i,x) | i{:}\, I \wedge x{:}\,\mathbb{S}_i\}$ with metric $d((i,x),(j,y)) \triangleq$ if $i = j$ then $d_i(x,y)$ else 1. The following hold: $\mathsf{P}(\coprod_{i:I} \mathbb{S}_i) \cong \prod_{i:I} \mathsf{P}(\mathbb{S}_i)$, i.e., a subset in the sum *is* a sum $\coprod_{i:I} S_i$ of subsets. Similarly, $\mathsf{C}(\coprod_{i:I} \mathbb{S}_i) \cong \prod_{i:I} \mathsf{C}(\mathbb{S}_i)$. Moreover, $(\coprod_{i:I} S_i)_\delta = \coprod_{i:I} (S_i)_\delta$ for $\delta \leq 1$.

Remark 1. A hybrid system on a Euclidean space $\mathbb{S}$ is a pair $\mathcal{H} = (F,G)$ of relations on $\mathbb{S}$, equivalently it is a subset $F + G$ of the metric space $\mathbb{S}^2 + \mathbb{S}^2$. Thus, closure and δ-fattening are applicable to hybrid systems and subsets of $\mathbb{S}$.

3 Analyses and Robustness

We identify analyses with arrows $A{:}\,\mathbf{Po}(X,Y)$ in the category **Po** of complete lattices and monotonic maps between them. The partial order $\leq$ allows to define over-approximations and compare them. We consider $\leq$ as an information order, thus: $x_0 \leq x$ means that x_0 is an over-approximation of x, $x_1 \leq x_0$ means that x_1 is a bigger over-approximation than x_0 (hence, less informative).

The complete lattice $\bot < \top$ of truth values, usually denoted Σ, is isomorphic to $\mathbb{P}(1)$ with 1 being the singleton set $\{\mathsf{fail}\}$, namely $\top$ (true) corresponds to $\emptyset$ (cannot fail), while $\bot$ (false) corresponds to $\{\mathsf{fail}\}$ (may fail). Safety analyses are arrows $A\colon \mathbf{Po}(X, \Sigma)$, and over-approximations may give false negatives.

Example 4. Safety analysis for transition systems on $\mathbb{S}$ corresponds to the arrow $\mathsf{Sf}\colon \mathbf{Po}(\mathbb{P}(\mathbb{S}^2) \times \mathbb{P}(\mathbb{S}) \times \mathbb{P}(\mathbb{S}), \Sigma)$ such that $\mathsf{Sf}(R, I, B) = \top \iff R^*(I)$ and B are disjoint, i.e., the set $R^*(I)$ of states reachable from the set I of initial states by (finitely many) R-transitions is disjoint from the set B of bad states.

Complete lattices do not have the structure to *quantify* imprecision. Thus, we restrict to complete lattices of the form $\mathbb{C}(\mathbb{S})$, with $\mathbb{S}$ a metric space, and use δ-fattening (Sec 2) to bound imprecision. Namely, given an over-approximation C' of $C\colon \mathbb{C}(\mathbb{S})$, i.e., $C \subseteq C'$ (or equivalently $C' \leq C$), we say that the imprecision of C' in over-approximating C is $\leq \delta \iff C \subseteq C' \subseteq C_\delta$.

For a metric space $\mathbb{S}$, there is an adjunction in **Po** (Galois connection) between $\mathbb{P}(\mathbb{S})$ and $\mathbb{C}(\mathbb{S})$. In particular, every $S\colon \mathbb{P}(\mathbb{S})$ has a *best over-approximation* $\overline{S}\colon \mathbb{C}(\mathbb{S})$. In other words, $\mathbb{C}(\mathbb{S})$ is an *abstract interpretation* of $\mathbb{P}(\mathbb{S})$ [1].

Definition 2 (Robustness [6]). *Given $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ with $\mathbb{S}_1$ and $\mathbb{S}_2$ metric spaces, we say that:*

- *A is **robust** at $C \iff \forall \epsilon > 0. \exists \delta > 0. A(C_\delta) \subseteq A(C)_\epsilon$.*
- *A is **robust** $\iff$ A is robust at every C.*

Robustness is a trivial property of analyses in a δ-discrete setting (Ex 1).

Proposition 1. *If $\mathbb{S}_1$ is δ-discrete, then every $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ is robust.*

Most analyses are not cast in the right form to ask whether they are robust, but usually one can show that they have the right form up to isomorphisms in **Po**.

Example 5. We consider analyses for (topological) transition systems [2].

1. Reachability $\mathsf{Rf}_R\colon \mathbf{Po}(\mathbb{P}(\mathbb{S}), \mathbb{P}(\mathbb{S}))$ for a transition system R on $\mathbb{S}$ is not a map on closed subsets, but can be replaced by the arrow $C \mapsto \overline{\mathsf{Rf}_R(C)}$ on $\mathbb{C}(\mathbb{S})$. This is the canonical way to turn arrows on $\mathbb{P}(\mathbb{S})$ into arrows on $\mathbb{C}(\mathbb{S})$, but it may fail to be idempotent. A better choice is the *best* idempotent arrow on $\mathbb{C}(\mathbb{S})$ over-approximating Rf_R, denoted Rs_R and called **safe reachability** in [6], i.e., $\mathsf{Rs}_R(C) \triangleq$ the smallest $C'\colon \mathbb{C}(\mathbb{S})$ such that $C \subseteq C'$ and $R(C') \subseteq C'$.
2. Reachability $\mathsf{Rf}\colon \mathbf{Po}(\mathbb{P}(\mathbb{S}^2) \times \mathbb{P}(\mathbb{S}), \mathbb{P}(\mathbb{S}))$ for transition systems on $\mathbb{S}$. First, we replace $\mathbb{P}(\mathbb{S}^2) \times \mathbb{P}(\mathbb{S})$ with the isomorphic $\mathbb{P}(\mathbb{S}^2 + \mathbb{S})$ (see Ex 3). Second, we proceed as done for Rf_R. In particular, we can replace Rf with safe reachability $\mathsf{Rs}\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}^2) \times \mathbb{C}(\mathbb{S}), \mathbb{C}(\mathbb{S}))$ for *closed* transition systems on $\mathbb{S}$.
3. Safety $\mathsf{Sf}\colon \mathbf{Po}(\mathbb{P}(\mathbb{S}^2) \times \mathbb{P}(\mathbb{S}) \times \mathbb{P}(\mathbb{S}), \Sigma)$ is definable in terms of reachability Rf, namely $\mathsf{Sf}(R, I, B) \iff \mathsf{Rf}(R, I) \# B$, where $\#$ is the disjointness predicate. Any replacement for Rf induces a corresponding notion of safety, e.g., safe safety $\mathsf{Ss}\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}^2) \times \mathbb{C}(\mathbb{S}) \times \mathbb{C}(\mathbb{S}), \Sigma)$ is $\mathsf{Ss}(R, I, B) \iff \mathsf{Rs}(R, I) \# B$.

4 Best Robust Approximations

Intuitively, when an analysis $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ is robust at C, $A(C)$ is *useful* also in the presence of small amounts of imprecision. This is obvious for analyses $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \Sigma)$, where robustness at C means $A(C_\delta) = A(C)$ when δ is *small*.

Definition 3. *Given* $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$*, we say that:*

- $A'\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ *is a robust approximation of* $A \iff$ A' *is robust and* $\forall C. A'(C) \leq A(C)$.
- $A^{\boxdot}\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ *is a* ***best robust approximation*** *of* $A \iff$ $A^{\boxdot}$ *is a robust approximation of* A *such that* $A'(C) \leq A^{\boxdot}(C)$ *for every robust approximation* A' *of* A *and* C.

When $\mathbb{S}_1$ is δ-discrete (i.e., $\exists \delta > 0. \forall x. B(x, \delta) = \{x\}$) every $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ is robust, thus $A^{\boxdot} = A$. When $\mathbb{S}_1$ is not δ-discrete, the following result ensures existence of best robust approximations.

Theorem 1. *If* $\mathbb{S}_2$ *is a compact metric space, then* $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ *has a best robust approximation* $A^{\boxdot}$ *given by* $A^{\boxdot}(C) = \bigcap\{A(C_\delta) | \delta > 0\}$.

Proof. If A' is a robust approximation of A, then $A'(C) \leq A^{\boxdot}(C)$. In fact

- $A'(C) =$ because $C' = \bigcap_\epsilon C'_\epsilon$ for every $C'\colon \mathbb{C}(\mathbb{S}_2)$
- $\bigcap_\epsilon A'(C)_\epsilon =$ because A' is robust
- $\bigcap_\delta A'(C_\delta) \leq$ because A' approximates A
- $\bigcap_\delta A(C_\delta) = A^{\boxdot}(C)$.

We now prove that $A^{\boxdot}$ is robust. $\mathbb{S}_2$ compact implies $\mathbb{C}(\mathbb{S}_2)$ continuous lattice and $C'_\epsilon \ll C'$ for $C'\colon \mathbb{C}(\mathbb{S}_2)$ and $\epsilon > 0$ (see [6, Appendix A.1]). When $C' = A^{\boxdot}(C)$ we have $\forall \epsilon > 0. \exists \delta > 0. C'_\epsilon \leq A(C_\delta)$, since $C'_\epsilon \ll C'$ and $\{A(C_\delta) | \delta > 0\}$ is directed. But $A(C_\delta) \leq A^{\boxdot}(C_{\delta'})$ when $\delta' < \delta$, thus $\forall \epsilon > 0. \exists \delta' > 0. C'_\epsilon \leq A^{\boxdot}(C_{\delta'})$. □

When $\mathbb{S}_1$ is not topologically **discrete**, i.e., $\forall x. \exists \delta > 0. B(x, \delta) = \{x\}$ fails, and $\mathbb{S}_2$ is not compact, there are $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ with no $A^{\boxdot}$.

Example 6. If $\mathbb{S}_1$ is not discrete, then there exists x and a sequence $(x_n | n)$ such that $\forall n. 0 < d_1(x_{n+1}, x) < d_1(x_n, x)/2$. If $\mathbb{S}_2$ is not compact, then there exists a sequence of distinct elements $(y_n | n)$ with no accumulation points, therefore any subset of $\{y_n | n\}$ is closed. We claim that the map $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$ such that $A(C) = \{y_n | \exists m. x_m \in C \wedge m \leq n\}$ has no best robust approximation. The proof is similar to that in [6, Ex 4.6].

The table below combines the results in [6,7] and in this paper to give an almost complete picture on existence or non-existence of best robust approximations for analyses $A\colon \mathbf{Po}(\mathbb{C}(\mathbb{S}_1), \mathbb{C}(\mathbb{S}_2))$, which depends only on properties of the metric spaces $\mathbb{S}_1$ and $\mathbb{S}_2$:

existence of $A^{\boxdot}$	$\mathbb{S}_2$ compact	$\mathbb{S}_2$ not compact
$\mathbb{S}_1$ δ-discrete	$\forall A. A^{\boxdot} = A$ (trivial case)	
$\mathbb{S}_1$ discrete but not δ-discrete	$\forall A. A^{\boxdot}$ exists	????
$\mathbb{S}_1$ not discrete		$\exists A. A^{\boxdot}$ does not exist

The only grey spot is when $\mathbb{S}_1$ is topologically discrete, but not δ-discrete.

$\mathcal{H}$	S_0	s	S_f	S_s	S_r	S_R	
$\mathcal{H}_E$	$[0,1]$	0	$[0]$	S_f	$\mathbf{S_0}$	S_0	
		$0 < s \leq 1$	$[s,1]$	S_f	S_f	S_f	
$\mathcal{H}_D$	$[0,1]$	0	S_0	S_0	S_0	S_0	
		$0 < s \leq 1$	$(0,s]$	$\mathbf{S_0}$	S_0	S_0	
$\mathcal{H}_T$	$\{(x,y) \mid 0 \leq x \leq y \leq 1\}$	$(0,1)$	$S^*(0)$	S_f	S_f	S_f	$b = 0$
		$(0,1)$	$S^*(b)$	$\mathbf{S_f \cup S(0)}$	S_s	S_s	$0 < b < 1$
		$(0,1)$	$S^*(1)$	S_f	S_f	$\mathbf{S_0}$	$b = 1$

For $\mathcal{H}_E$ and $\mathcal{H}_D$ we take $\mathcal{H}_0 = (F_0, G_0)$ with $F_0 = [0,1] \times [-1,1]$ and $G_0 = [0,1]^2$.
For $\mathcal{H}_T = (F, G)$ we take $\mathcal{H}_0 = (\overline{F}, G_0)$ with $G_0 = \{(y,y) | y\colon [0,1]\} \times \{(0,y) | y\colon [0,1]\}$, and we use the notation $S(b) \triangleq [0,b] \times [b]$ and $S^*(b) \triangleq \cup_n S(b^n)$ for subsets of S_0, where $b\colon [0,1]$. In the limit cases $b = 0, 1$ one has $S^*(0) = S(1) \cup S(0)$ and $S^*(1) = S(1)$.
The differences in the approximations of the reachable states are highlighted in **bold**.

Table 1. Safe and robust over-approximations of the set of reachable states.

5 Examples

Finally, we compare different reachability analyses for three hybrid systems:

$\mathcal{H}_E$ a quantity x grows according to ODE $\dot{x} = x$ when $0 \leq x < 1$, and stays constant when it reaches the threshold 1, i.e., $\dot{x} = 0$ when $x = 1$.

$\mathcal{H}_D$ a quantity x decreases according to ODE $\dot{x} = -x$ when $0 < x \leq 1$, and it is *instantaneously* reset to 1 when it is 0, i.e., $x^+ = 1$ when $x = 0$.

$\mathcal{H}_T$ a timer x grows while the timeout y stays constant, i.e., $\dot{x} = 1 \& \dot{y} = 0$ when $0 \leq x < y \leq 1$, when x reaches y it is reset and the timeout updated, i.e., $x^+ = 0 \& y^+ = by$ when $0 < x = y \leq 1$ (with b constant in the interval $[0,1]$), moreover $x^+ = 0 \& y^+ = 1$ when $0 = x = y \leq 1$, i.e., y is reset to 1.

Table 1 gives for each $\mathcal{H}$ above (and initial state s) the following sets:

- $S_f \triangleq \mathsf{Rf}_{\mathcal{H}}(s)$ set of states reachable (from s) in finitely many transitions, S_f is always a subset of the set S of the states reachable in finite time;
- $S_s \triangleq \mathsf{Rs}_{\mathcal{H}}(s)$ superset of S computed by safe reachability;
- $S_r \triangleq \mathsf{Rs}^{\boxdot}_{\mathcal{H}}(s)$ superset of S_s robust w.r.t. over-approximations of s;
- $S_R \triangleq \mathsf{Rs}^{\boxdot}(\overline{\mathcal{H}}, s)$ superset of S_s robust w.r.t. over-approximations of $\overline{\mathcal{H}}$ & s.

Note that S_r depends on a compact subset S_0 (including s and the *support* of $\mathcal{H}$), and S_R depends also on a compact hybrid system $\mathcal{H}_0$ (with support S_0 and over-approximating $\mathcal{H}$). In particular, $\mathcal{H}_0$ constrains the over-approximations of $\mathcal{H}$. The inclusions $[s \in] S_f \subseteq S_s \subseteq S_r \subseteq S_R [\subseteq S_0]$ hold always, Table 1 shows that any of the inclusions can be either strict or an equality.

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