A note on SDT in Filter Spaces
Working Draft

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Introduction

This work was motivated by the desire to integrate Evaluation Logic EL_T (see [Mog93]) and Synthetic Domain Theory SDT (see [Hyl91]). A first step in this direction was to consider several computational monads over the category **Cpo** of predomains, and check whether they preserve regular monos – in **Cpo** regular monos correspond to inductive subsets. The dominion of regular monos (in a left exact category) is the simplest candidate for a LCF-like logic, where assertions are Horn sequents with equalities among the atomic formulas. Although, most computational monads over **Cpo** are unproblematic, there are two important exceptions:

- continuations $\Sigma^{(\Sigma^X)}$, where Σ is the cpo classifying open subsets
- Plotkin's powerdomain $P_p(X_\perp)$, where $P_p(X)$ is the free binary semi-lattice over X

(similar problems arise with the other powerdomains).

Proposition 0.1 *In* **Cpo** *exists a regular mono m s.t. Tm is not monic, when T is the monad of continuations or a powerdomain.*

Proof Let $L \cong (1+L)_\perp$ be the domain of lazy natural numbers, whose elements are: $s^n(0)$, $s^n(\perp)$ and ∞. The order on L is generated by $s^n(\perp) < s^{n+1}(\perp), s^n(0), \infty$ for every $n \in N$. m is the inductive subset of maximal elements, i.e. the equaliser of $s, s' : L \to L$, where $s(s^n(\perp)) = s^{n+1}(\perp)$, $s(s^n(0)) = s'(s^n(0)) = s^{n+1}(0)$ and are the identity otherwise. ∎

To overcome these problems, we have decided to replace **Cpo** with the category **Rep** of replete objects in some model $(\mathcal{E}, \Sigma)$ of SDT (see [Tay91]). This note establishes some basic facts about replete objects in the category **Fil** of filter spaces (see [Hyl79]), namely:

- **SFP** $\subset$ **Rep**, where $\subset$ means full subcategory (see Theorem 2.4)
- a characterisation of the regular subobjects (in **Rep**) of an SFP (see Theorem 2.7)
- the following axiom is valid in **Fil**: $m: \underline{X} \to \underline{Y}$ regular mono implies $\Sigma(m)$ regular epi, provided $\underline{Y}$ is a topological space (see Theorem 2.9).

1 Filter spaces: basic definitions facts

Basic definitions (see [Hyl79]):

- F is a filter over X iff is a non empty collections of nonempty subsets of X s.t. $U \subseteq V \subseteq X, U \in F \supset V \in F$ and $U, V \in F \supset (U \cap V) \in F$
- F is a filter base over X iff is a non empty collections of nonempty subsets of X s.t. $U, V \in F \supset \exists W \in F.W \subseteq (U \cap V)$

  if F is a filter base, then $[F] \triangleq \{U \subseteq X | \exists V \in G.V \subseteq U\}$

- $\mathcal{F}(X)$ is the collection of filters over X
- $\underline{X} = (X, F_X)$ is a filter space iff $F_X : X \to \mathcal{F}(X)$,
  $F \subseteq G \in \mathcal{F}(X), F \in F_X(x) \supset G \in F_X(x)$ and
  $[\{x\}] \in F_X(x)$

  we write $F \downarrow_X x$ for $[F] \in F_X(x)$
- $f : \underline{X} \to \underline{Y}$ is continuous iff $f : X \to Y$ and
  $F \downarrow_X x \supset f(F) \stackrel{\Delta}{=} \{f(U) | U \in F\} \downarrow_Y fx$
- **Fil** is the category of filter spaces and continuous functions.

Basic properties (see [Hyl79]):

- **Fil** is a quasi-topos (see [Wyl76])
- **Top** is a full reflective subcategory of **Fil**

  the embedding $\Delta : \mathbf{Top} \hookrightarrow \mathbf{Fil}$ maps a topological space (X, τ_X) to the filter space over X s.t. $F \in F_X(x) \stackrel{\Delta}{\Longleftrightarrow} \{U \in \tau_X | x \in U\} \subseteq F$

  the left adjoint T to Δ maps a filter space $\underline{X}$ to X with the induced topology $O \in \tau_X \stackrel{\Delta}{\Longleftrightarrow} \forall x \in O. \forall F \in F_X(x). O \in F$

  Δ preserves coproducts and exponentials, but it does not preserve coequalisers.

Remark 1.1 Several categories can be viewed as full subcategories of **Top**, e.g.: **Set**, **PoSet**, **PreOrd**, **Cpo** (both ω-cpos and D-cpos). By abuse of notation, Δ will denote also the embedding into **Fil** of these full subcategories of **Top**.

$$\begin{array}{ccccc}
\mathbf{Set} & \hookrightarrow & \mathbf{PoSet} \subset \mathbf{PreOrd} & & \\
\big\downarrow & & \big\downarrow & & \\
\mathbf{Cpo} & \hookrightarrow & \mathbf{Top} & \hookrightarrow & \mathbf{Fil}
\end{array}$$

We say that a filter space $\underline{X}$ is a set/poset/... iff it is the image via Δ of a set/poset/....

Further properties:

- The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$, which maps (X, τ_X) to the underlying set X, is faithful and has left and right adjoints $\Delta \dashv U \dashv \Gamma$ that are full and faithful:
  - $\Delta : \mathbf{Set} \hookrightarrow \mathbf{Top}$ maps X to X with the discrete topology $\mathcal{P}(X)$
  - $\Gamma : \mathbf{Set} \hookrightarrow \mathbf{Top}$ maps X to X with the chaotic topology $\{\emptyset, X\}$

  moreover Δ is left exact, but it does not preserve infinite products

  The forgetful functor $U : \mathbf{Fil} \to \mathbf{Set}$ enjoys similar properties
- In **Set** the only non trivial dominance is 2
- in **PoSet** and **Cpo** there are two dominances: 2 classifies decidable subobjects and Σ classifies open/closed subobjects
- in **PreOrd**, **Top** and **Fil** there are three dominances: 2 classifies decidable subobjects, Σ classifies open/closed subobjects and Ω classifies regular subobjects.

  in Fil there are six non isomorphic filter structures over 2: 2, Σ, Σ', Ω, Ω' and Ω''. I expect that only three of them are dominances.

Proposition 1.2 *For any of the embeddings $\Delta : \mathcal{C}_1 \to \mathcal{C}_2$, if $\Sigma \in \mathcal{C}_1$ is a dominance and $\underline{X} \in \mathcal{C}_1$, then the set of Σ-subobjects of $\underline{X}$ in $\mathcal{C}_1$ is isomorphic (via Δ) to the set of Σ-subobjects of $\underline{X}$ in $\mathcal{C}_2$.*

2 Replete objects in filter spaces

In [Tay91] a model of SDT is a pair $(\mathcal{E}, \Sigma)$, where $\mathcal{E}$ is a topos and Σ is a dominance satisfying certain axioms. Here we work in the setting of a quasi-topos, and we write Ω for the strong subobjects classifier. We investigate the model $(\mathbf{Fil}, \Sigma)$, where Ω is the filter space over 2 s.t. every filter converges to every point, while Σ is the filter space over 2 s.t. $F \downarrow 0$ iff F is a filter over X and $F \downarrow 1$ iff $F = [\{1\}]$.

- there is a bijection between $\mathbf{Fil}(\underline{X}, \Sigma)$ and the set τ_X of open subsets of $\underline{X}$, where O open in $\underline{X}$ $\overset{\Delta}{\iff} \forall x \in O.\forall F \in F_X(x).O \in F$, given by $f \mapsto O_f \overset{\Delta}{=} \{x \in X | f(x) = 1\}$.

  $F \downarrow f$ in $\Sigma^{\underline{X}} \overset{\Delta}{\iff}$
  $\forall x.\forall G \in F_X(x).\exists W \in F.\exists U \in G.W(U) \subseteq \uparrow (fx) \iff$ by definition of Σ
  $\forall x \in O_f.\forall G \in F_X(x).\exists W \in F.\exists U \in G.W(U) = \{1\} \iff$ by definition of $W(U)$
  $\forall x \in O_f.\forall G \in F_X(x).\exists W \in F.\exists U \in G.U \subseteq \cap\{O_g | g \in W\} \iff$ by G filter
  $\forall x \in O_f.\forall G \in F_X(x).\exists W \in F.(\cap\{O_g | g \in W\}) \in G$

- The bijection $f \mapsto O_f$ is an isomorphism from $\Sigma^{\underline{X}}$ to the filter space $\Sigma(\underline{X})$ over τ_X s.t. $F \downarrow O \overset{\Delta}{\iff} \forall x \in O.\forall G \in F_X(x).\exists W \in F.(\cap W) \in G$.

- The lifting induced by Σ maps $\underline{X}$ to the filter space $\underline{X}_\perp$ over $X_\perp$ s.t. $F \downarrow \perp$ always and $F \downarrow x$ in $\underline{X}_\perp \overset{\Delta}{\iff} (F \cap \mathcal{F}(X)) \downarrow x$ in $\underline{X}$.

Proposition 2.1 *If $\underline{X}$ is a preorder, i.e. $F \downarrow x$ in $\underline{X}$ iff $\uparrow \{x\} \in F$, then $F \downarrow U$ in $\Sigma(\underline{X})$ iff $\forall D \subseteq_{fin} U.\{V \in \tau_X | D \subseteq V\} \in F$ and $\Sigma(\underline{X})$ is the set τ_X of upward closed subsets of X with the Scott topology, whose base is given by the collection of all $\uparrow \{\uparrow D\} \overset{\Delta}{=} \{U \in \tau_X | D \subseteq U\}$. with $D \subseteq_{fin} X$.*

Corollary 2.2 *If $\underline{X}$ is a finite preorder, then $\Sigma(\underline{X})$ is the finite poset $(\tau_X, \subseteq)$, where τ_X is the set of upward closed subsets of X. If $\underline{X}$ is a set, then $\Sigma(\underline{X})$ is $\mathcal{P}(X)$ with the Scott topology, whose base is given by the collection of all $\uparrow \{\uparrow D\} \overset{\Delta}{=} \{U \subseteq X | D \subseteq U\}$ with $D \subseteq_{fin} X$.*

From the above results, one can easily prove the following principles (see [Tay91]):

- Phoa's principle: Σ^Σ is isomorphic to $\{(x, y) \in \Sigma^2 | x \supset y\}$, moreover $\Sigma^\Sigma \cong \Sigma_\perp$
- Markov's principle: every $\phi \in \Sigma$ is $\neg\neg$-closed, indeed $\neg\neg(\phi) = \phi$ for every $\phi \in \Omega$
- Scott's principle: if $\Phi \in \Sigma(\Sigma(\underline{N}))$ and $N \in \Phi$, then exists $m \in N$ s.t. $[0, m) \in \Phi$

  the natural number object $\underline{N}$ in $\mathbf{Fil}$ is $\Delta(N)$
- Σ is closed w.r.t. finite meets
- Σ is closed w.r.t. countable joints, i.e. $\exists\colon \Sigma(\underline{N}) \to \Sigma$ is continuous, where $\exists(D) = 1$ iff $D \neq \emptyset$.

In [Tay91] the following facts about the category **Rep** of replete objects are stated:

I assume that they hold also in the setting of a quasi-topos

- **Rep** is a full reflective subcategory of **Fil**, therefore **Rep** has all limits and they are computed like in **Fil**,
- the reflection preserves finite products, therefore $\underline{Y}^{\underline{X}}$ is replete when $\underline{Y}$ is,
- $\underline{Y}_\perp$ is replete when $\underline{Y}$ is.

Proposition 2.3 *Every finite poset is replete.*

Proof It is enough to prove that $\underline{X}$ is the equaliser of two continuous maps between powers of Σ. Let $f, g: \Sigma(\Delta X) \to \Sigma(\Sigma(\Delta X)) \times \Sigma(\Delta X)$ be the monotonic maps (between finite posets) s.t.

- $f(U) = \langle \{V \subseteq X | \exists x \in U.(\uparrow \{x\}) \subseteq V\}, \downarrow U\rangle$
- $g(U) = \langle \{V \subseteq X | \forall x.U \subseteq (\downarrow \{x\}) \supset (\uparrow \{x\}) \subseteq V\}, U\rangle$

We claim that $m: \underline{X} \hookrightarrow \Sigma(\Delta X)$ s.t. $m(x) = \downarrow \{x\} \stackrel{\Delta}{=} \{y \in X | y \leq x\}$ is the equaliser of f and g. In fact, m preserves and reflect the order. Moreover, $U = (\downarrow \{x\})$ implies $f(U) = g(U) = \langle \{V \subseteq X | (\uparrow \{x\}) \subseteq V\}, (\downarrow \{x\})\rangle$, and conversely $f(U) = g(U)$ implies $U = (\downarrow U)$ and U must have a maximum element, otherwise X/U is in the first component of $g(U)$ but not of $f(U)$. ∎

I don't think a similar argument is applicable to other cpos, because the first component of g is not continuous in general.

Theorem 2.4 (SFP are replete) *Every SFP is replete.*

Proof If $\underline{X}$ is SFP, then it is the limit of an ω^{op}-chain of projections between finite posets. Since this limits are preserved by the embedding of **Cpo** into **Top** and the embedding of **Top** into **Fil** preserves all limits, then SFP are replete because replete objects are closed w.r.t. limits. ∎

I guess that all constructions on SFP that are induced by constructions on finite posets (e.g. finite products and coproducts, exponentials) are preserved by the embedding of SFP into Fil, in particular this is true for the construction of the Plotkin's powerdomain (because of the results in [Tay91, TP90]).

Lemma 2.5 (Regular monos in Rep) *Given $\underline{Y}$ and $m: \underline{X} \to \underline{Y}$ in* **Rep***, then m is a regular mono in* **Rep** *iff it is the equaliser of two maps from $\underline{Y}$ to $\Sigma(\Delta Z)$ for some set Z.*

Proof Obviously, the equaliser of two maps from $\underline{Y}$ to $\Sigma(\Delta Z)$ is regular in **Rep**, because $\Sigma(\Delta Z)$ is replete. The other direction is a consequence of the following chain of implications:

- m is the equaliser of two parallel maps into some replete object $\underline{Z}$
- m is the equaliser of two parallel maps into some power $\Sigma(\underline{Z})$, because $\underline{Z}$ replete implies $\epsilon_Z: \underline{Z} \to \Sigma(\Sigma(\underline{Z}))$ monic (see [Tay91]), where $\epsilon_Z(x) = (\lambda f: \Sigma^{\underline{Z}}.fx)$
- m is the equaliser of two parallel maps into some power $\Sigma(\Delta Z)$, because $\Sigma(i_Z): \Sigma(\underline{Z}) \to \Sigma(\Delta Z)$ is monic, where $i_Z: \Delta Z \to \underline{Z}$ is the identity on the underlying set.

∎

Remark 2.6 A similar result holds also in the category **Cpo** of cpos/D-cpos, namely: $m: \underline{X} \to \underline{Y}$ is a regular mono in **Cpo** iff it is the equaliser of two maps from $\underline{Y}$ to $\Sigma(\Delta Z)$ for some set Z. In fact, $\epsilon_Z: \underline{Z} \to \Sigma(\Sigma(\underline{Z}))$ and $\Sigma(i_Z): \Sigma(\underline{Z}) \to \Sigma(\Delta Z)$ are always monic. One can give even a concrete characterisation of the regular subobjects of $\underline{Y}$ in **Cpo**: they are the inductive subsets of $\underline{Y}$ with the induced order.

Theorem 2.7 (Characterisation regular subobjects of SFP) *In* **Rep** *the regular subobjects of an SFP $\underline{Y}$ are the inductive subsets on $\underline{Y}$ with the topology induced by the Scott topology on $\underline{Y}$.*

this characterisation applies to any cpo $\underline{X}$ which is in Rep

Proof Given $m: \underline{X} \to \underline{Y}$, the claim is a consequence of the following chain of equivalences:

- m is a regular mono in **Rep**
- m is the equaliser in **Rep**/**Fil** of two parallel maps $f, g: \underline{Y} \to \Sigma(\Delta Z)$ for some set Z, by the previous theorem

- m is the equaliser in **Top** of two parallel maps $f, g: \underline{Y} \to \Sigma(\Delta Z)$ for some set Z, because $\underline{Y}$ and $\Sigma(\Delta Z)$ are in **Top** and the embedding of **Top** into **Fil** preserves limits.

Finally, one has to prove (by analogy with the concrete characterisation of the regular subobjects in **Cpo**) that given a D-cpo $\underline{Y}$ and $m: \underline{X} \hookrightarrow \underline{Y}$ in **Top**, m is the equaliser of two maps from $\underline{Y}$ to $\Sigma(\Delta Z)$ for some set Z iff $\underline{X}$ is an inductive subset of $\underline{Y}$ with the topology induced by τ_Y. In fact, $\Sigma(\Delta Z)$ is a D-cpo (by Corollary 2.2). ∎

Probably Rep is not a full (reflective) subcategory of Top, although both Top and Rep are reflective subcategories of Fil containing Σ (and closed under isomorphisms). In fact, Rep is the smallest one among *internal* full reflective subcategories of Fil. In the context of cartesian closed categories, *internal* full reflective subcategory means (externally speaking) that the reflection preserves products, or equivalently that the subcategory is an exponential ideal. However, the reflection of Fil into Top does not preserve products (see [Hyl79]).

Lemma 2.8 (Main Lemma) *If $m: \underline{X} \hookrightarrow \underline{Y}$ is a regular mono in* **Fil** *and $\underline{Y}$ is a topological space, then $e = \Sigma(m)$ is a regular epi in* **Fil**.

Proof If $\underline{X}$ is a topological space, we write $\tau_X(x)$ for the set $\{U \in \tau_X | x \in U\}$ of neighbours of x. We have that $F \downarrow x$ in $\underline{Y} \iff \forall U \in \tau_Y(x).U \in F$, because $\underline{Y}$ is a topological space.
Since in **Fil** the regular subobjects of a topological space $\underline{Y}$ are exactly its subspaces, we can assume that m is the inclusion $X \subseteq Y$, and $F \downarrow x$ in $\underline{X} \iff \forall U \in \tau_Y(x).U \cap X \in F$. In what follows we use the adjunction $(\_)_L \dashv (\_)_R$ between τ_Y and τ_X, where $U_L = U \cap X$, i.e. $\Sigma(m)(U)$, and V_R is the biggest open set in τ_Y s.t. $(V_R)_L = V$.
We know already that $e = \Sigma(m)$ is epic, since $e: \tau_Y \to \tau_X$ is surjective. Therefore, to prove that e is a regular epi in **Fil**, it is enough to show that $G \downarrow V$ in $\Sigma(\underline{X})$ implies $V = e(V')$ and $G = e(G')$ for some $G' \downarrow V'$ in $\Sigma(\underline{Y})$. More precisely, we show that for some suitable V', which depends on V and G, one can take $G' = G_R \stackrel{\Delta}{=} [\{W_R | W \in G\}]$, where $W_R \stackrel{\Delta}{=} \{V_R | V \in W\} \subseteq \tau_Y$.

- $G \downarrow V$ in $\Sigma(\underline{X}) \stackrel{\Delta}{\iff}$
- $\forall x \in V. \forall F \downarrow x. \exists W \in G. (\cap W) \in F \iff$ since $\underline{X}$ is a subspace of $\underline{Y}$
- $\forall x \in V. \exists U \in \tau_Y(x). \exists W \in G. U_L \subseteq (\cap W)$

Let $U: V \to \tau_Y$ and $W: V \to G$ be choice functions s.t. $\forall x \in V. x \in (Ux)_L \subseteq \cap(Wx)$. One can assume w.l.o.g. that $Ux \subseteq V_R$, otherwise U may be replaced by $U'x \stackrel{\Delta}{=} (Ux) \cap V_R$. We now show that $V' \stackrel{\Delta}{=} \cup\{Ux | x \in V\} \in \tau_Y$ is s.t. $G_R \downarrow V'$ in $\Sigma(\underline{Y})$ (it is immediate to show that $V'_L = V$):

- $G_R \downarrow V'$ in $\Sigma(\underline{Y}) \iff$ since $\underline{Y}$ is a topological space
- $\forall y \in V'. \exists U \in \tau_Y(y). \exists W \in G_R. U \subseteq (\cap W) \iff$ by definition of V' and G_R
- $\forall x \in V. \forall y \in Ux. \exists U \in \tau_Y(y). \exists W \in G. U \subseteq (\cap W_R)$
  given $x \in V$ and $y \in Ux$, $U \subseteq (\cap W_R)$ is satisfied by taking $U = Ux$ and $W = Wx$, because
  $(Ux)_L \subseteq \cap Wx$ by definition of $U: V \to \tau_Y$ and $W: V \to G$
  $Ux \subseteq \cap(Wx)_R$ by the universal property of the adjunction $(\_)_L \dashv (\_)_R$.

∎

Theorem 2.9 (Main Theorem) *if $\underline{Y}$ is a topological space and $m: \underline{X} \to \underline{Y}$ is a regular mono in* **Fil**, *then $\Sigma(m)$ is a regular epi in* **Fil** *and $\Sigma^2(m)$ is a regular mono in* **Rep**.

Proof $\Sigma(m)$ is a regular epi in **Fil** (by Lemma 2.8). Since any contravariant functor of the form $FX = A^X$ maps colimits to limits and $\Sigma(\_)$ maps filter spaces to replete objects, then $\Sigma(\_)$ maps coequalisers in **Fil** to equalises in **Rep**. ∎

Topological spaces are not closed under $\Sigma^2(\_)$, but SFP are.

Corollary 2.10 $\Sigma^2(\_)$ *preserves regular subobjects (in* **Rep***) of SFPs.*

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