
ON THE SATISFIABILITY OF LOCAL FIRST-ORDER LOGICS WITH DATA

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ABSTRACT. We study first-order logic over unordered structures whose elements carry a finite number of data values from an infinite domain. Data values can be compared wrt. equality. As the satisfiability problem for this logic is undecidable in general, we introduce a family of local fragments. They restrict quantification to the neighbourhood of a given reference point that is bounded by some radius. Our first main result establishes decidability of the satisfiability problem for the local radius-1 fragment in presence of one “diagonal relation”. On the other hand, extending the radius leads to undecidability. In a second part, we provide the precise decidability and complexity landscape of the satisfiability problem for the existential fragments of local logic, which are parameterized by the number of data values carried by each element and the radius of the considered neighbourhoods. Altogether, we draw a landscape of formalisms that are suitable for the specification of systems with data and open up new avenues for future research.

1. INTRODUCTION

Data logics have been introduced to reason about structures whose elements are labeled with a value from an infinite alphabet [32]. The idea is to extend classic mathematical structures by a mapping that associates with every element of the universe a value from an infinite domain. When comparing data values only for equality, this view is equivalent to extending the underlying signature by a binary relation symbol whose interpretation is restricted to an equivalence relation. Among potential applications are XML reasoning or the specification of concurrent systems and distributed algorithms. Expressive decidable fragments include notably two-variable logics over data words and data trees [5, 6]. The decidability frontier is fragile, though. Extensions to two data values, for example, quickly lead to an undecidable satisfiability problem. From a modeling point of view, those extensions still play an important role. When specifying the *input-output behavior* of distributed algorithms [18, 28], processes get an input value and produce an output value, which requires two data values per process. In leader election or renaming algorithms, for instance, a process gets its unique identifier as input, and it should eventually output the identifier of a common leader (leader election) or a unique identifier from a restricted name space (renaming).

Key words and phrases: First Order Logic with Data, Satisfiability Problem.

In this paper, we consider a natural extension of first-order logic over unordered structures whose elements carry two or more data values from an infinite domain. There are two major differences between most existing formalisms and our language. While previous data logics are usually interpreted over words or trees, we consider unordered structures (or multisets). When each element of such a structure represents a process, we therefore do not assume a particular processes architecture, but rather consider clouds of computing units. Moreover, decidable data logics are usually limited to one value per element, which would not be sufficient to model an input-output relation. Hence, our models are algebraic structures consisting of a universe and functions assigning to each element a tuple of integers. We remark that, for many distributed algorithms, the precise data values are not relevant, but whether or not they are the same is. Like [5, 6], we thus add binary relations that allow us to test if two data values are identical and, for example, whether all output values were already present in the collection of input values (as required for leader election).

The first fundamental question that arises is whether a given specification is consistent. This leads us to the satisfiability problem. While the general logic considered here turns out to be undecidable already in several restricted settings, we show that interesting fragments preserve decidability. The fragments we consider are *local* logics in the sense that data values can only be compared within the neighborhood of a (quantified) reference element of the universe. In other words, comparisons of data values are restricted to elements whose distance to some reference point is bounded by a given radius.

We first consider a fragment over structures with two data values where the first value at the reference point can be compared with any second value in the neighborhood in terms of what we call the *diagonal relation*. In particular, we do not allow the symmetrical relation. However, we do not restrict comparisons of first values with each other in a neighborhood, nor do we restrict comparisons of second values with each other. Note that adding only one diagonal relation still constitutes an extension of the (decidable) two-variable first-order logic with two equivalence relations [23–25]: equivalence classes consist of those elements with the same first value, respectively, second value. In fact, our main technical contribution is a reduction to this two-variable logic. The reduction requires a careful relabelling of the underlying structures so as to be able to express the diagonal relation in terms of the two equivalence relations. In addition, the reduction takes care of the fact that our logic does not restrict the number of variables. We can actually count elements up to some threshold and express, for instance, that at most five processes crash (in the context of distributed algorithms). This is a priori not possible in two-variable logic. We also show that this local fragment has an undecidable satisfiability problem for any radius strictly greater than 1.

In a second part, we study orthogonal local fragments where global quantification is restricted to being existential (while quantification inside a local property is still unrestricted). We obtain decidability for (i) radius 1 and an arbitrary number of data values, and for (ii) radius 2 and two data values. In all cases, we provide tight complexity upper and lower bounds. These results mark the exact decidability frontier of the existential fragments: satisfiability is undecidable as soon as we consider radius 3 in presence of two data values, or radius 2 together with three data values.

Related Work. As already mentioned, data logics over word and tree structures were studied in [5, 6]. In particular, the authors showed that two-variable first-order logic on words has a decidable satisfiability problem. Other types of data logics allow *two* data values to be associated with an element [23, 25], though they do not assume a linearly ordered

or tree-like universe. Again, satisfiability turned out to be decidable for the two-variable fragment of first-order logic.

Orthogonal extensions for multiple data values include shuffle expressions for nested data [3] and temporal logics [13, 22]. Other generalizations of data logics allow for an order on data values [29, 33]. The application of formal methods in the context of distributed algorithms is a rather recent but promising approach (cf. for a survey [26]). A particular branch is the area of parameterized systems, which, rather than on data, focuses on the (unbounded) number of processes as the parameter [4, 15]. Other related work includes [14], which considers temporal logics involving quantification over processes but without data, while [1] introduces an (undecidable) variant of propositional dynamic logic that allows one to reason about totally ordered process identifiers in ring architectures. First-order logics for *synthesizing* distributed algorithms were considered in [7, 19]. A counting extension of two-variable first-order logic over finite data words with one data value per position has been studied in [2].

Outline. The paper is structured as follows. In Section 2, we recall important notions such as structures and first-order logic, and we introduce local first-order logic. Section 3 presents decidability and undecidability results for our first fragment. In Section 4, we study the existential fragments and present their decidability frontier as well as complexity results. We conclude in Section 5.

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2. STRUCTURES AND FIRST-ORDER LOGIC

2.1. Data Structures.

We define here the class of models we are interested in. It consists of sets of nodes containing data values with the assumption that each node is labeled by a set of predicates and carries the same number of values. We consider hence Σ a finite set of unary relation symbols (sometimes called unary predicates) and an integer $Da \geq 0$. A *Da-data structure* over Σ is a tuple $\mathfrak{A} = (A, (P_\sigma)_{\sigma \in \Sigma}, f_1, \dots, f_{Da})$ (in the following, we simply write $(A, (P_\sigma), f_1, \dots, f_{Da})$) where A is a nonempty finite set, $P_\sigma \subseteq A$ for all $\sigma \in \Sigma$, and f_i s are mappings $A \rightarrow \mathbb{N}$. Intuitively A represents the set of nodes and $f_i(a)$ is the i -th data value carried by a for each node $a \in A$. For $X \subseteq A$, we let $Val_{\mathfrak{A}}(X) = \{f_i(a) \mid a \in X, i \in \{1, \dots, Da\}\}$. The set of all Da -data structures over Σ is denoted by $\text{Data}[Da, \Sigma]$.

While this representation is often very convenient to deal with data values, a more standard way of representing mathematical structures is in terms of binary relations. For every $(i, j) \in \{1, \dots, Da\} \times \{1, \dots, Da\}$, the mappings $f_1, \dots, f_{Da}$ determine a binary relation $i \sim_j^{\mathfrak{A}} \subseteq A \times A$ as follows: $a \sim_j^{\mathfrak{A}} b$ iff $f_i(a) = f_j(b)$. We may omit the superscript $\mathfrak{A}$ if it is clear from the context and if $Da = 1$, as there will be only one relation, we may write $\sim$ for $1 \sim_1$.

2.2. First-Order Logic.

Let $Da \geq 0$ be an integer, $\Gamma \subseteq \{i \sim_j \mid i, j \in \{1, \dots, Da\}\}$ a set of binary relation symbols, which determines the binary relation symbols at our disposal, and $\mathcal{V} = \{x, y, \dots\}$ a countably infinite set of variables. The set $\text{dFO}[Da, \Sigma, \Gamma]$ of first-order formulas interpreted over Da -data structures over Σ is inductively given by the grammar:

$$\varphi ::= \sigma(x) \mid x \sim y \mid x = y \mid \varphi \vee \varphi \mid \neg \varphi \mid \exists x. \varphi$$

where x and y range over $\mathcal{V}$, σ ranges over Σ , and $\sim \in \Gamma$. We use standard abbreviations such as $\wedge$ for conjunction and $\Rightarrow$ for implication. We write $\varphi(x_1, \dots, x_k)$ to indicate that the free variables of φ are among $x_1, \dots, x_k$. We call φ a *sentence* if it does not contain free variables. Moreover, we use Γ_{Da} to represent the set of binary relation symbols $\{i \sim_j \mid i, j \in \{1, \dots, Da\}\}$.

For $\mathfrak{A} = (A, (P_\sigma), f_1, \dots, f_{Da}) \in \text{Data}[Da, \Sigma]$ and a formula $\varphi \in \text{dFO}[Da, \Sigma, \Gamma]$, the satisfaction relation $\mathfrak{A} \models_I \varphi$ is defined wrt. an interpretation function $I : \mathcal{V} \rightarrow A$. The purpose of I is to assign an interpretation to every (free) variable of φ so that φ can be assigned a truth value. For $x \in \mathcal{V}$ and $a \in A$, the interpretation function $I[x/a]$ maps x to a and coincides with I on all other variables. We then define:

$$\begin{aligned} \mathfrak{A} \models_I \sigma(x) & \text{ if } I(x) \in P_\sigma \\ \mathfrak{A} \models_I \varphi_1 \vee \varphi_2 & \text{ if } \mathfrak{A} \models_I \varphi_1 \text{ or } \mathfrak{A} \models_I \varphi_2 \\ \mathfrak{A} \models_I x \sim_j y & \text{ if } I(x) \sim_j^{\mathfrak{A}} I(y) \\ \mathfrak{A} \models_I \neg \varphi & \text{ if } \mathfrak{A} \not\models_I \varphi \\ \mathfrak{A} \models_I x = y & \text{ if } I(x) = I(y) \\ \mathfrak{A} \models_I \exists x. \varphi & \text{ if there is } a \in A \text{ s.t. } \mathfrak{A} \models_{I[x/a]} \varphi \end{aligned}$$

Finally, for a data structure $\mathfrak{A} = (A, (P_\sigma), f_1, \dots, f_{Da})$, a formula $\varphi(x_1, \dots, x_k)$ and $a_1, \dots, a_k \in A$, we write $\mathfrak{A} \models \varphi(a_1, \dots, a_k)$ if there exists an interpretation function I such that $\mathfrak{A} \models_{I[x_1/a_1] \dots [x_k/a_k]} \varphi$. In particular, for a sentence φ , we write $\mathfrak{A} \models \varphi$ if there exists an interpretation function I such that $\mathfrak{A} \models_I \varphi$.

Example 1. Assume a unary predicate $\text{leader} \in \Sigma$. The following formula from $\text{dFO}[2, \Sigma, \Gamma_2]$ expresses correctness of a leader-election algorithm: (i) there is a unique process that has been elected leader, and (ii) all processes agree, in terms of their output values (their second data), on the identity (the first data) of the leader: $\exists x. (\text{leader}(x) \wedge \forall y. (\text{leader}(y) \Rightarrow y = x)) \wedge \forall y. \exists x. (\text{leader}(x) \wedge x \sim_1 y)$. We assume here that for each element represents a process, the first data is used to characterize its input value and the second data its output value.

Note that every choice of Γ gives rise to a particular logic, whose formulas are interpreted over data structures over Σ . We will focus on the satisfiability problem for these logics. Let $\mathcal{F}$ denote a generic class of first-order formulas, parameterized by D , Σ and Γ . In particular, for $\mathcal{F} = \text{dFO}$, we have that $\mathcal{F}[Da, \Sigma, \Gamma]$ is the class $\text{dFO}[Da, \Sigma, \Gamma]$. The satisfiability problem for $\mathcal{F}$ wrt. data structures is defined as follows:

DATASAT($\mathcal{F}, Da, \Gamma$)	
Input:	A finite set Σ and a sentence $\varphi \in \mathcal{F}[Da, \Sigma, \Gamma]$.
Question:	Is there $\mathfrak{A} \in \text{Data}[Da, \Sigma]$ such that $\mathfrak{A} \models \varphi$?

The following negative result (see [21, Theorem 1]) calls for restrictions of the general logic.

Theorem 1. [21] *The problem $\text{DATASAT}(\text{dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ is undecidable, even when we require that $\Sigma = \emptyset$.*

2.3. Local First-Order Logic.

We are interested in logics that combine the advantages of the logics considered so far, while preserving decidability. With this in mind, we will study *local* logics, where the scope of quantification is restricted to the view (or neighborhood) of a given element.

The view of an element a includes all elements whose distance to a is bounded by a given radius. It is formalized using the notion of a Gaifman graph (for an introduction, see [27]). In fact, we use a variant that is suitable for our setting and that we call *data graph*. Fix sets Σ and Γ . Given a data structure $\mathfrak{A} = (A, (P_\sigma), f_1, \dots, f_{Da}) \in \text{Data}[Da, \Sigma]$, we define its *data graph* $\mathcal{G}(\mathfrak{A}) = (V_{\mathcal{G}(\mathfrak{A})}, E_{\mathcal{G}(\mathfrak{A})})$ with set of vertices $V_{\mathcal{G}(\mathfrak{A})} = A \times \{1, \dots, Da\}$ and set of edges $E_{\mathcal{G}(\mathfrak{A})} = \{((a, i), (b, j)) \in V_{\mathcal{G}(\mathfrak{A})} \times V_{\mathcal{G}(\mathfrak{A})} \mid a = b \text{ and } i \neq j, \text{ or } i \sim_j \in \Gamma \text{ and } a \sim_j b\}$. Note that the definition of the edges of this graph depends of the set Γ , however to simplify the figures we do not always write it and make sure it will always be clear from the context. Figures 1(A) and 2(A) provide examples of the graph $\mathcal{G}(\mathfrak{A})$ for two data structure with 2 data values. In the first case, the set of considered binary relation symbols is $\{1 \sim_1, 2 \sim_2, 1 \sim_2\}$ whereas in the second case it is $\Gamma_2 = \{1 \sim_1, 2 \sim_2, 1 \sim_2, 2 \sim_1\}$. We point out that the graph $\mathcal{G}(\mathfrak{A})$ is directed and indeed we see in the first Figure 1(A) that since $2 \sim_1$ is not considered, we have some unidirectional edges which we depict dashed.

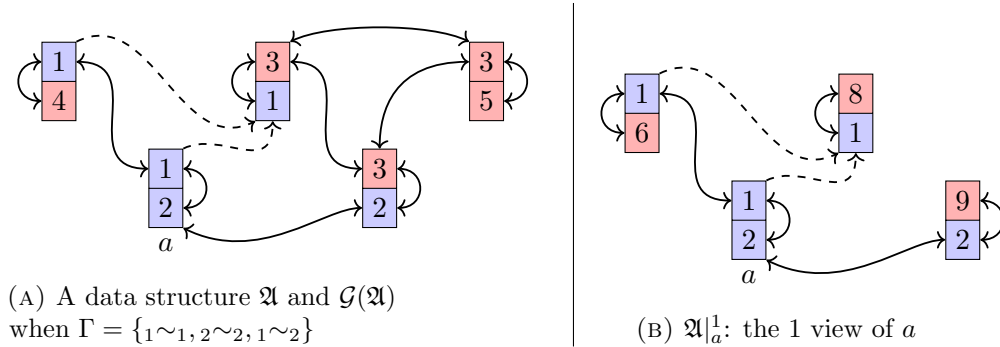


FIGURE 1

We then define the distance $d^{\mathfrak{A}}((a, i), (b, j)) \in \mathbb{N} \cup \{\infty\}$ between two elements (a, i) and (b, j) from $A \times \{1, \dots, Da\}$ as the length of the shortest *directed* path from (a, i) to (b, j) in $\mathcal{G}(\mathfrak{A})$. In fact, as the graph is directed, the distance function might not be symmetric. For $a \in A$ and $r \in \mathbb{N}$, the *radius- r -ball* around a is the set $B_r^{\mathfrak{A}}(a) = \{(b, j) \in V_{\mathcal{G}(\mathfrak{A})} \mid d^{\mathfrak{A}}((a, i), (b, j)) \leq r \text{ for some } i \in \{1, \dots, Da\}\}$. This ball contains the elements of $V_{\mathcal{G}(\mathfrak{A})}$ that can be reached from $(a, 1), \dots, (a, Da)$ through a directed path of length at most r . On Figure 1(A) the blue nodes represent $B_1^{\mathfrak{A}}(a)$ and on Figure 2(A) they represent $B_2^{\mathfrak{A}}(a)$.

Let $f_{\text{new}} : A \times \{1, \dots, Da\} \rightarrow \mathbb{N} \setminus \text{Val}_{\mathfrak{A}}(A)$ be an injective mapping. The *r -view* of a in $\mathfrak{A}$ is the structure $\mathfrak{A}|_a^r = (A', (P'_\sigma), f'_1, \dots, f'_n) \in \text{Data}[Da, \Sigma]$. Its universe is $A' = \{b \in A \mid (b, i) \in B_r^{\mathfrak{A}}(a) \text{ for some } i \in \{1, \dots, Da\}\}$. Moreover, $f'_i(b) = f_i(b)$ if $(b, i) \in B_r^{\mathfrak{A}}(a)$, and $f'_i(b) = f_{\text{new}}((b, i))$ otherwise. Finally, P'_σ is the restriction of P_σ to A' . To illustrate this

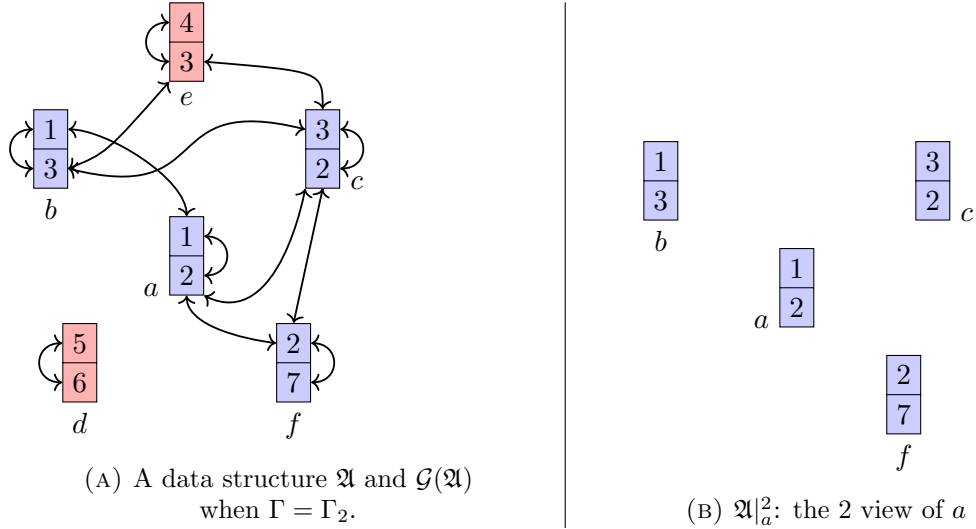


FIGURE 2

definition, we use again Figures 1 and 2. On Figure 1(B), the structure $\mathfrak{A}|_a^1$ is given by the four elements that contain at least one blue node. However, the values of the red nodes have to be replaced by pairwise distinct fresh values not contained in $\{1, \dots, 5\}$. Note that the precise values do not matter. On Figure 2(B), the structure $\mathfrak{A}|_a^2$ is depicted, in that case there is no need of fresh value.

We are now ready to present the logic $r\text{-Loc-dFO}[Da, \Sigma, \Gamma]$, where $r \in \mathbb{N}$, interpreted over structures from $\text{Data}[Da, \Sigma]$. It is given by the grammar

$$\varphi ::= \langle\!\langle \psi \rangle\!\rangle_x^r \mid x = y \mid \exists x. \varphi \mid \varphi \vee \varphi \mid \neg \varphi$$

where ψ is a formula from $\text{dFO}[Da, \Sigma, \Gamma]$ with (at most) one free variable x . This logic uses the *local modality* $\langle\!\langle \psi \rangle\!\rangle_x^r$ to specify that the formula ψ should be interpreted over the r -view of the element associated to the variable x . For $\mathfrak{A} \in \text{Data}[Da, \Sigma]$ and an interpretation function I , we have $\mathfrak{A} \models_I \langle\!\langle \psi \rangle\!\rangle_x^r$ iff $\mathfrak{A}|_{I(x)}^r \models_I \psi$.

Example 2. Let us illustrate what can be specified by formulas in $1\text{-Loc-dFO}[2, \Sigma, \Gamma_2]$. We can rewrite the formula from Example 1 so that it falls into our fragment as follows: $\exists x. (\langle\!\langle \text{leader}(x) \rangle\!\rangle_x^1 \wedge \forall y. (\langle\!\langle \text{leader}(y) \rangle\!\rangle_y^1 \Rightarrow x = y)) \wedge \forall y. \langle\!\langle \exists x. \text{leader}(x) \wedge y \sim_1 x \rangle\!\rangle_y^1$. Indeed note that in the formula of Example 1, when we write $\forall y. \exists x. (\text{leader}(x) \wedge x \sim_1 y)$, then for all elements a , the element labelled with *leader* has to appear in the 1-view of a since they respectively share the same value on their second and first value. The next formula specifies an algorithm in which all processes suggest a value and choose a new value among those that have been suggested at least twice: $\forall x. \langle\!\langle \exists y. \exists z. y \neq z \wedge x \sim_1 y \wedge x \sim_1 z \rangle\!\rangle_x^1$. We can also specify partial renaming, i.e., two output values agree only if their input values are the same: $\forall x. \langle\!\langle \forall y. (x \sim_2 y \Rightarrow x \sim_1 y) \rangle\!\rangle_x^1$. Conversely, $\forall x. \langle\!\langle \forall y. (x \sim_1 y \Rightarrow x \sim_2 y) \rangle\!\rangle_x^1$ specifies partial fusion of equivalences classes.

2.4. Previously Known Results.

We list in this subsection previously known results related to the satisfiability problem we are studying and that we shall rely on.

2.4.1. 0 and 1 Data Value. First, note that formulas in $\text{dFO}[0, \Sigma, \emptyset]$ (i.e. where no data is considered) correspond to first order logic formulas with a set of predicates and equality test as a unique relation. As mentioned in Chapter 6.2.1 of [11], these formulas belong to the *Löwenheim class with equality* also called as the relational monadic formulas, and their satisfiability problem is in NEXP. Furthermore, thanks to [16] (Theorem 11), we know that this latter problem is NEXP-hard even if one considers formulas which use only two variables.

Theorem 2. *[11, 16] $\text{DATASAT}(\text{dFO}, 0, \emptyset)$ is NEXP-complete.*

In [30], the authors study the satisfiability problem for Hybrid logic over Kripke structures where the transition relation is an equivalence relation, and they show that it is N2EXP-complete. Furthermore in [17], it is shown that Hybrid logic can be translated to first-order logic in polynomial time and this holds as well for the converse translation. Since 1-data structures can be interpreted as Kripke structures with one equivalence relation, altogether this allows us to obtain the following preliminary result about the satisfiability problem of $\text{dFO}[1, \Sigma, \{1 \sim 1\}]$.

Theorem 3. *[17, 30] $\text{DATASAT}(\text{dFO}, 1, \{1 \sim 1\})$ is N2EXP-complete.*

2.4.2. A Normal Form. When $\Gamma = \emptyset$, satisfiability of monadic first-order logic is decidable [12, Corollary 6.2.2] and the logic actually has a useful normal form. Let $\varphi(x_1, \dots, x_n, y) \in \text{dFO}[Da, \Sigma, \emptyset]$ (note that since the set of binary relation symbols is empty, the number Da of data values does not matter here) and $k \geq 1$ be a natural number. We use $\exists^{\geq k} y. \varphi(x_1, \dots, x_n, y)$ as an abbreviation for $\exists y_1 \dots \exists y_k. \bigwedge_{1 \leq i < j \leq k} \neg(y_i = y_j) \wedge \bigwedge_{1 \leq i \leq k} \varphi(x_1, \dots, x_n, y_i)$. Thus, $\exists^{\geq k} y. \varphi$ says that there are at least k distinct elements y that verify φ . We call a formula of the form $\exists^{\geq k} y. \varphi$ a *threshold formula*. We also use $\exists^{\leq k} y. \varphi$ as an abbreviation for $\exists^{\geq k} y. \varphi \wedge \neg \exists^{\geq k+1} y. \varphi$.

When $\Gamma = \emptyset$, the out-degree of every element is 0 so that, over this particular signature, we deal with structures of bounded degree. The following lemma will turn out to be useful. It is due to Hanf's locality theorem [20, 27] for structures of bounded degree (cf. [8, Theorem 2.4]).

Lemma 1. *Every formula from $\text{dFO}[Da, \Sigma, \emptyset]$ with one free variable x is effectively equivalent to a Boolean combination of formulas of the form $\sigma(x)$ with $\sigma \in \Sigma$ and threshold formulas of the form $\exists^{\geq k} y. \varphi_U(y)$ where $U \subseteq \Sigma$ and $\varphi_U(y) = \bigwedge_{\sigma \in U} \sigma(y) \wedge \bigwedge_{\sigma \in \Sigma \setminus U} \neg \sigma(y)$.*

2.4.3. Extended Two-Variable First-Order Logic. One way to skirt the undecidability result of Theorem 1 is to restrict the number of allowed variables in the formulas. We denote by $\text{dFO}^2[Da, \Sigma, \Gamma]$ the two-variable fragment of $\text{dFO}[Da, \Sigma, \Gamma]$, i.e. the set of formulas in $\text{dFO}[Da, \Sigma, \Gamma]$ which use only two variables (usually x and y). In a two-variable formula, however, each of the two variables can be used arbitrarily often. The satisfiability problem of two-variable logic over arbitrary finite structures with two equivalence relations is decidable [25, Theorem 15]. By a straightforward reduction to this problem, we obtain :

Theorem 4. [25] *The problem $\text{DATASAT}(\text{dFO}^2, 2, \{1 \sim_1, 2 \sim_2\})$ is decidable.*

Actually, this result can be generalized to *extended* two-variable first-order logic. A formula belongs to $\text{ext-dFO}^2[Da, \Sigma, \Gamma]$ if it is of the form $\varphi \wedge \psi$ where $\varphi \in \text{dFO}[Da, \Sigma, \emptyset]$ and $\psi \in \text{dFO}^2[Da, \Sigma, \Gamma]$. To obtain the next result, the idea consists in first translating the formula $\varphi \in \text{dFO}[Da, \Sigma, \emptyset]$ to a two-variable formula thanks to new unary predicates.

Proposition 1. *The problem $\text{DATASAT}(\text{ext-dFO}^2, 2, \{1 \sim_1, 2 \sim_2\})$ is decidable.*

Proof. We first show that one can reduce the first-order part with $\Gamma = \emptyset$ to a two-variable formula. Let φ be a sentence in $\text{dFO}[Da, \Sigma, \emptyset]$. We show we can effectively construct $\varphi' \in \text{dFO}^2[Da, \Sigma', \emptyset]$ with $\Sigma \subseteq \Sigma'$ such that φ is satisfiable iff φ' is satisfiable. Furthermore, if a structure $\mathfrak{A}$ satisfies φ , then we can add an interpretation of the predicates in $\Sigma' \setminus \Sigma$ to $\mathfrak{A}$ to get a model for φ' . Conversely, if a structure $\mathfrak{A}'$ satisfies φ' , then forgetting the interpretation of the predicates in $\Sigma' \setminus \Sigma$ in $\mathfrak{A}'$ give us a model for φ .

We apply Lemma 1 to φ and then obtain φ'' . As there is no free variable in φ , the formula φ'' is a boolean combination of formulas of the form $\exists^{\geq k} y. \varphi_U(y)$ where $U \subseteq \Sigma$. Let M be the maximal such k (if there is no threshold formula, φ'' is either *true* or *false*). We define a set of unary predicates $\Lambda_M = \{\eta_i \mid 1 \leq i \leq M\}$ and let $\Sigma' = \Sigma \cup \Lambda_M$. Intuitively the unary predicates in Λ_M will be used to count up to M the elements labeled by the same predicates in Σ associating each element of the domain with a unique numerical index. The following formulas will specify the meaning of the elements of Λ_M . First, let $\varphi_{\text{same}}(x, y) = \bigwedge_{\sigma \in \Sigma} \sigma(x) \leftrightarrow \sigma(y)$. With this, we define:

$$\begin{aligned} \varphi_\eta^1 &:= \forall x. \bigvee_{i \in [1, M]} (\eta_i(x) \wedge \bigwedge_{j \in [1, M] \setminus \{i\}} \neg \eta_j(x)) \\ \varphi_\eta^2 &:= \forall x. \bigwedge_{i \in [1, M-1]} (\eta_i(x) \rightarrow \neg \exists y. (x \neq y \wedge \varphi_{\text{same}}(x, y) \wedge \eta_i(y))) \\ \varphi_\eta^3 &:= \forall x. \bigwedge_{i \in [2, M]} (\eta_i(x) \rightarrow (\exists y. \varphi_{\text{same}}(x, y) \wedge \eta_{i-1}(y))) \end{aligned}$$

We then denote $\varphi_\eta := \varphi_\eta^1 \wedge \varphi_\eta^2 \wedge \varphi_\eta^3 \in \text{dFO}^2[Da, \Sigma', \emptyset]$. We provide an informal explanation of the previous formulas:

- the formula φ_η^1 ensures that each element is labeled by a single label from Λ_M ;
- the formula φ_η^2 ensures that there are no two elements labeled by the same predicates in Σ which are labeled by the same element η_i with $i \leq M-1$;
- the formula φ_η^3 ensures that if an element is labeled with η_i with $i \geq 2$ then there must exist another element with the same labels in Σ labeled by η_{i-1} .

Then, for a model $\mathfrak{A} \in \text{Data}[Da, \Sigma']$ of φ_η with carrier set A , an element $a \in A$, and an integer $1 \leq i \leq M$, we have that $a \in P_{\eta_i}$ iff $|\{b \in A \mid \text{for all } \sigma \in \Sigma, a \in P_\sigma \text{ iff } b \in P_\sigma\}| \geq i$. Then in φ'' , we replace all threshold formulas $\exists^{\geq k} y. \varphi_U(y)$ with $\exists y. \varphi_U(y) \wedge \eta_k(y)$ in order to obtain $\varphi''' \in \text{dFO}^2[Da, \Sigma', \emptyset]$. Finally we take φ' as $\varphi''' \wedge \varphi_\eta$.

We can now proceed with the proof of the Proposition. Let $\varphi \wedge \psi$ be a sentence such that $\varphi \in \text{dFO}[Da, \Sigma, \emptyset]$ and $\psi \in \text{dFO}^2[Da, \Sigma, \Gamma]$. We determine $\Sigma' \supseteq \Sigma$ and φ' in $\text{dFO}^2[Da, \Sigma, \Gamma]$ according to what we said above. Then, by Theorem 4, it only remains to show that $\varphi \wedge \psi$ is satisfiable iff $\varphi' \wedge \psi$ is satisfiable.

Suppose there is $\mathfrak{A} \in \text{Data}[Da, \Sigma]$ such that $\mathfrak{A} \models \varphi \wedge \psi$. According to what we said, we can add propositions from $\Sigma' \setminus \Sigma$ to $\mathfrak{A}$ to get a data structure $\mathfrak{A}'$ such that $\mathfrak{A}' \models \varphi'$. As ψ does not speak about propositions in $\Sigma' \setminus \Sigma$, we have $\mathfrak{A}' \models \psi$ and, therefore, $\mathfrak{A}' \models \varphi' \wedge \psi$.

Conversely, let $\mathfrak{A}' \in \text{Data}[Da, \Sigma']$ such that $\mathfrak{A}' \models \varphi' \wedge \psi$. Then, “forgetting” in $\mathfrak{A}'$ all labels in $\Sigma' \setminus \Sigma$ yields a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \varphi$. As we still have $\mathfrak{A} \models \psi$, we conclude $\mathfrak{A} \models \varphi \wedge \psi$. $\square$

3. DEALING WITH TWO DATA VALUES

3.1. Decidability with Radius One and One Diagonal Relation.

We will show in this section that, when considering structures with two data values, $\text{DATASAT}(1\text{-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2, 1 \sim_2\})$ (or, symmetrically, $\text{DATASAT}(1\text{-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2, 2 \sim_1\})$), i.e. the local fragment of radius 1 with a set of binary symbols restricted to $\{1 \sim_1, 2 \sim_2, 1 \sim_2\}$ is decidable. To this end, we will give a reduction to the problem $\text{DATASAT}(\text{ext-dFO}^2, 2, \{1 \sim_1, 2 \sim_2\})$ and rely on Proposition 1. Before to give the formal details of this reduction, we present here the main steps it is based on:

- We transform a formula $\varphi \in 1\text{-Loc-dFO}[2, \Sigma, \Gamma]$, into a formula $\chi \in \text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathbf{C}_M, \emptyset]$ such that φ is satisfiable iff χ has a well-typed model. In a well-typed model, each element a is labelled with some unary predicates in $\mathbf{C}_M$ specifying for each subset R of binary relations in Γ the number (up to a constant M) of elements sharing the same unary predicates in Σ and in relation with a for all relations in R . For instance, such a labelling can indicate that there are two elements having the same first data value as a . To perform this translation, we rely on the form of formula in $1\text{-Loc-dFO}[2, \Sigma, \Gamma]$ and use Lemma 1 (eq is a special predicate to detect elements with the same two values).
- We then need to ensure that a given model is well-typed. We can achieve this by adding predicates to count up to a constant M (we label a single element with 1, a single element with 2, etc) and then use a formula with two variables to check the correct labelling. We could indeed build a formula in $\text{dFO}^2[2, \Sigma \cup \mathbf{C}_M \cup \{\text{eq}\} \cup \Lambda_M, \Gamma]$ (where Λ_M are extra unary predicates to count elements with the same properties) to verify if a model is a well typed. The problem is that to rely on the result of Proposition 1, we need the formula not to use the diagonal relation $1 \sim_2$.
- To get rid of the diagonal relations, we show that we can add for each value v in our model, an extra element that we label with a special predicate ed and that have the value v in its two fields. We then demonstrate how to ensure diagonal relations by making a detour via these extra elements. We call this extended model well-diagonalized and provide a two variable formula in $\text{dFO}^2[2, \Sigma \cup \mathbf{C}_M \cup \{\text{eq}, \text{ed}\} \cup \Lambda_M, \Gamma_{df}]$ to check that a model is well diagonalized.

We now move to the formal reduction. Henceforth, we fix a finite set Σ as well as $\Gamma = \{1 \sim_1, 2 \sim_2, 1 \sim_2\}$ and we let the *diagonal-free set* be $\Gamma_{df} = \{1 \sim_1, 2 \sim_2\}$. Moreover, we let Θ range over arbitrary finite sets such that $\Sigma \subseteq \Theta$ and $\Theta \cap \{\text{eq}, \text{ed}\} = \emptyset$, where eq and ed are special unary symbols that are introduced below.

We start with some crucial notion. Suppose $\Gamma' \subseteq \Gamma$ (which will later be instantiated by either Γ_{df} or Γ). Consider a data structure $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Theta]$ with $\Sigma \subseteq \Theta$.

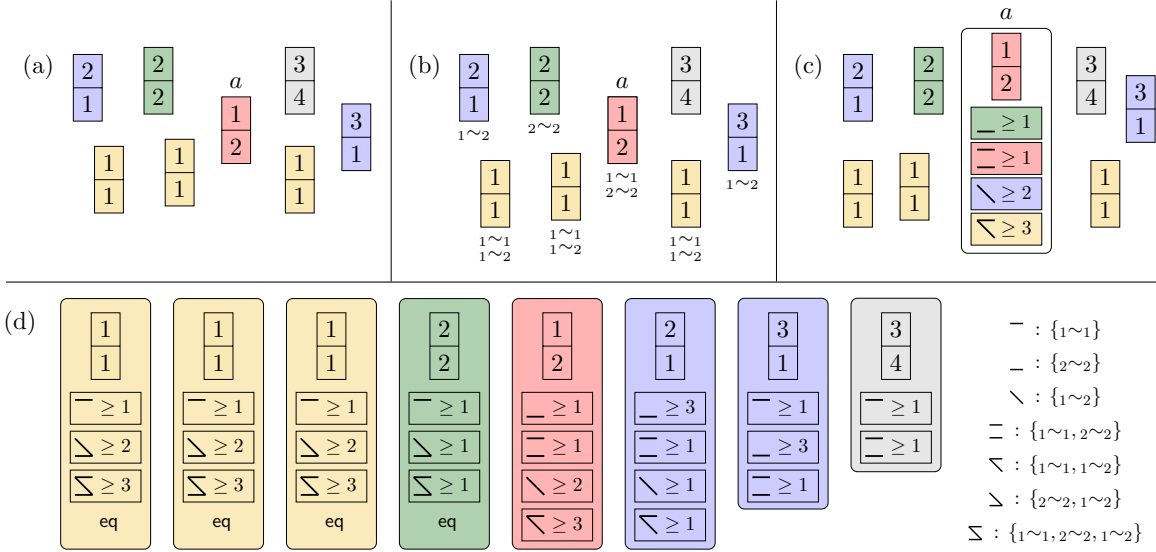


FIGURE 3. (a) A data structure over $\Sigma = \emptyset$. (b) Adding unary predicates for a given element a . (c) Adding counting constraints to a . (d) A well-typed data structure from $\text{Data}[\{\text{eq}\} \cup C_3]$.

Given $U \subseteq \Sigma$ and a nonempty set $R \subseteq \Gamma'$, the *environment* of $a \in A$ is defined as

$$\text{Env}_{\mathfrak{A}, \Sigma, \Gamma'}(a, U, R) = \{b \in A \mid U = \{\sigma \in \Sigma \mid b \in P_\sigma\} \text{ and } R = \{i \sim_j \in \Gamma' \mid a \sim_j^{\mathfrak{A}} b\}\}$$

Thus, it contains the elements that carry exactly the labels from U (relative to Σ) and to which a is related precisely in terms of the relations in R (relative to Γ').

Example 3. Consider $\mathfrak{A} \in \text{Data}[2, \Sigma]$ from Figure 3(a) where $\Sigma = \emptyset$. Then, the set $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, \{1 \sim_1, 1 \sim_2\}) = \text{Env}_{\mathfrak{A}, \Sigma, \Gamma_{df}}(a, \emptyset, \{1 \sim_1\})$ contains exactly the yellow elements (with data-value pairs $(1, 1)$), and $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, \{1 \sim_2\})$ contains the two blue elements (with data-value pairs $(2, 1)$ and $(3, 1)$).

Let us now go through the reduction step by step.

Step 1: Transform Binary into Unary Relations

In the first step, we get rid of the binary relations by representing them as unary ones. In fact, in a formula $\langle\langle \psi \rangle\rangle_x^1$ from $1\text{-Loc-dFO}[2, \Sigma, \Gamma]$, ψ only talks about elements that are directly related to $a = I(x)$ in terms of pairs from Γ . We can hence rewrite ψ into ψ' so that all comparisons are wrt. x , i.e., they are of the form $x \sim_j y$. Then, a pair $i \sim_j \in \Gamma$ can be seen as a unary predicate that holds at b iff $a \sim_j b$. In this way, we eliminate the binary relations and replace ψ' with a first-order formula ψ'' over unary predicates.

Example 4. Adding unary relations to a data structure for a given element a is illustrated in Figure 3(b) (recall that $\Sigma = \emptyset$). We can see for instance that we add to the yellow nodes the labels $1 \sim_1$ and $1 \sim_2$ as the first value of a is equal to the first and second values of these nodes.

Thanks to the unary predicates, we can now apply Lemma 1 (which was a consequence of locality of first-order logic over unary symbols only). That is, to know whether ψ'' holds

when x is interpreted as a , it is enough to know how often every unary predicate is present in the environment of a , counted only up to some $M \geq 1$. However, we will then give up the information of whether the two data values at a coincide or not. Therefore, we introduce a unary predicate eq , which shall label those events whose two data values coincide. Accordingly, we say that $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$ is *eq-respecting* if, for all $a \in A$, we have $a \in P_{\text{eq}}$ iff $f_1(a) = f_2(a)$.

Once we add this information to a , it is enough to know the size of $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R)$ for every $U \subseteq \Sigma$ and nonempty $R \subseteq \Gamma$, measured up to M . To reason about these sizes, we introduce a unary predicate $\{U, R, m\}$ for all $U \subseteq \Sigma$, nonempty sets $R \subseteq \Gamma$, and $m \in \{1, \dots, M\}$ (which is interpreted as “ $\geq m$ ”). We also call such a predicate a *counting constraint* and denote the set of all counting constraints by $\mathcal{C}_M$ (recall that we fixed Σ and Γ). For a finite set Θ with $\Sigma \subseteq \Theta$, we call $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[\Theta \cup \mathcal{C}_M]$ *cc-respecting* if, for all $a \in A$, we have $a \in P_{\{U, R, m\}}$ iff $|\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R)| \geq m$.

Finally, we call $\mathfrak{A} \in \text{Data}[2, \Theta \cup \{\text{eq}\} \cup \mathcal{C}_M]$ *well-typed* if it is eq-respecting and cc-respecting.

Example 5. In Figure 3(c), where we suppose $M = 3$ and $\Sigma = \emptyset$, the element a satisfies the counting constraints $\{\emptyset, \{2 \sim 2\}, 1\}$, $\{\emptyset, \{1 \sim 1, 2 \sim 2\}, 1\}$, $\{\emptyset, \{1 \sim 2\}, 2\}$, and $\{\emptyset, \{1 \sim 1, 1 \sim 2\}, 3\}$, as well as all inherited constraints for smaller constants (which we omitted). We write $\{U, R, m\}$ as $R \geq m$. In fact, pairs from R are represented as black bars in the obvious way (cf. Figure 3(d)); moreover, for each constraint, the corresponding elements have the same color. For instance, a satisfies the counting constraint $\{\emptyset, \{1 \sim 1, 1 \sim 2\}, 3\}$ as they are at least three nodes (the yellow ones) having as first and second values the same value as the first one of a . Finally, the data structure from Figure 3(d) is well-typed, i.e., eq- and cc-respecting. Again, we omit inherited constraints.

To summarize, we have the following reduction:

Lemma 2. For each formula $\varphi \in 1\text{-Loc-dFO}[2, \Sigma, \Gamma]$, we can effectively compute $M \in \mathbb{N}$ and $\chi \in \text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M, \emptyset]$ such that φ is satisfiable iff χ has a well-typed model.

Proof. Consider $\langle\langle \psi \rangle\rangle_x^1$ where ψ is a formula from $\text{dFO}[2, \Sigma, \Gamma]$ with one free variable x . Wlog., we assume that x is not quantified in ψ . We replace, in ψ , every occurrence of a formula $y \text{ } i \sim_j z$ with $y \neq x$ by

$$\bigvee_{\substack{k \in \{1, 2\} \\ k \sim_i, k \sim_j \in \Gamma}} x \text{ } k \sim_i y \wedge x \text{ } k \sim_j z.$$

Call the resulting formula ψ' . Replace, in ψ' , every formula $x \text{ } i \sim_j y$ by $i \sim_j(y)$ to obtain an $\text{dFO}[2, \Sigma \cup \Gamma, \emptyset]$ formula ψ'' . Consider $\mathfrak{A} = (A, (P_\sigma)_{\sigma \in \Sigma \cup \Gamma}, f_1, f_2,) \in \text{Data}[2, \Sigma \cup \Gamma]$ and an interpretation function I such that, for all $b \in A$ and $i \sim_j \in \Gamma$, we have $b \in P_{i \sim_j}$ iff $I(x) \text{ } i \sim_j b$. Then,

$$\mathfrak{A}|_{I(x)}^1 \models_I \psi(x) \iff \mathfrak{A}|_{I(x)}^1 \models_I \psi'(x) \iff \mathfrak{A}|_{I(x)}^1 \models_I \psi''(x).$$

According to Lemma 1, we can effectively transform ψ'' into an equivalent $\text{dFO}[2, \Sigma \cup \Gamma, \emptyset]$ formula $\hat{\psi}''$ that is a Boolean combination of formulas of the form $\sigma(x)$ with $\sigma \in \Sigma \cup \Gamma$ and threshold formulas of the form $\exists^{\geq k} y. \varphi_U(y)$ where $U \subseteq \Sigma \cup \Gamma$ and $\varphi_U(y) = \bigwedge_{\sigma \in U} \sigma(y) \wedge \bigwedge_{\sigma \in (\Sigma \cup \Gamma) \setminus U} \neg \sigma(y)$. Let M be the maximal such k (or $M = 0$ if there is no threshold formula). Again, we assume that x is not quantified in $\hat{\psi}''$.

We obtain the $\text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M, \emptyset]$ formula χ' from $\hat{\psi}''$ by replacing

- $1 \sim_2(x)$ by $\text{eq}(x)$, and $1 \sim_1(x)$ and $2 \sim_2(x)$ by true ,
- $\exists^{\geq k} y. \varphi_U(y)$ by $\begin{cases} \text{false} & \text{if } U \cap \Gamma = \emptyset \\ \wr U \cap \Sigma, U \cap \Gamma, k \wr(x) & \text{if } U \cap \Gamma \neq \emptyset \end{cases}$

We can then eliminate redundant *true* and *false*. Suppose a well-typed data structure $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Sigma \cup \Gamma \cup \{\text{eq}\} \cup \mathbf{C}_M]$ and an interpretation function I such that, for all $b \in A$ and $i \sim_j \in \Gamma$, we have $b \in P_{i \sim_j}$ iff $I(x) i \sim_j b$. Then,

$$\mathfrak{A}|_{I(x)}^1 \models_I \hat{\psi}''(x) \iff \mathfrak{A}|_{I(x)}^1 \models_I \chi'(x).$$

Moreover, for $U \subseteq \Sigma$, a nonempty set $R \subseteq \Gamma$, and $k \in \mathbb{N}$, we have

$$\mathfrak{A}|_{I(x)}^1 \models_I \wr U, R, k \wr(x) \iff \mathfrak{A} \models_I \wr U, R, k \wr(x).$$

We deduce that, for all well-typed $\mathfrak{A} \in \text{Data}[2, \Sigma \cup \Gamma \cup \{\text{eq}\} \cup \mathbf{C}_M]$ and interpretation functions I ,

$$\mathfrak{A} \models_I \langle\langle \psi \rangle\rangle_x^1 \iff \mathfrak{A} \models_I \chi'(x).$$

We can apply this translation to all the terms $\langle\langle \psi \rangle\rangle_x^1$ present in the formula $\varphi \in \text{1-Loc-dFO}[2, \Sigma, \Gamma]$ to obtain a formula $\chi \in \text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathbf{C}_M, \emptyset]$. From what we have seen if χ has a well-typed model then φ is satisfiable (note to obtain a data structure in $\text{Data}[2, \Sigma]$ from a well-typed model in $\text{Data}[2, \Sigma \cup \Gamma \cup \{\text{eq}\} \cup \mathbf{C}_M]$, we simply forget the unary predicates not in Σ). Furthermore from a data structure in $\text{Data}[2, \Sigma]$, we can easily build a well-typed model in $\text{Data}[2, \Sigma \cup \Gamma \cup \{\text{eq}\} \cup \mathbf{C}_M]$ (by respecting the typing rules) and this allows to conclude that if φ is satisfiable then χ has a well-typed model. $\square$

Step 2: Well-Diagonalized Structures

In $\mathbf{C}_M$, we still have the diagonal relation $(1, 2) \in \Gamma$. Our goal is to get rid of it so that we only deal with the diagonal-free set $\Gamma_{df} = \{1 \sim_1, 2 \sim_2\}$. The idea is again to extend a given structure $\mathfrak{A}$, but now we add new elements, one for each value $n \in \text{Val}_{\mathfrak{A}}(A)$, which we tag with a unary symbol *ed* and whose two data values are n . Diagonal equality will be ensured through making a detour via these ‘diagonal’ elements (hence the name *ed*).

Formally, when we start from some $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$, the data structure $\mathfrak{A} + \text{ed} \in \text{Data}[2, \Theta \cup \{\text{eq}, \text{ed}\}]$ is defined as $(A', (P'_\sigma), f'_1, f'_2)$ where $A' = A \uplus \text{Val}_{\mathfrak{A}}(A)$, $f'_i(a) = f_i(a)$ for all $a \in A$ and $i \in \{1, 2\}$, $f'_1(a) = f'_2(a) = a$ for all $a \in \text{Val}_{\mathfrak{A}}(A)$, $P'_\sigma = P_\sigma$ for all $\sigma \in \Theta \setminus \{\text{eq}\}$, $P'_{\text{ed}} = \text{Val}_{\mathfrak{A}}(A)$, and $P'_{\text{eq}} = P_{\text{eq}} \cup \text{Val}_{\mathfrak{A}}(A)$.

Example 6. The structure $\mathfrak{A} + \text{ed}$ is illustrated in Figure 4(a), with $\Theta = \emptyset$.

With this, we say that $\mathfrak{B} \in \text{Data}[2, \Theta \cup \{\text{eq}, \text{ed}\}]$ is *well-diagonalized* if it is of the form $\mathfrak{A} + \text{ed}$ for some *eq-respecting* $\mathfrak{A} \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$. Note that then $\mathfrak{B}$ is *eq-respecting*, too.

Example 7. The data structure $\mathfrak{A} + \text{ed}$ from Figure 4(a) is well-diagonalized. The one from Figure 4(b) is not well-diagonalized (in particular, it is not *eq-respecting*).

We will need a way to ensure that the considered data structures are well-diagonalized. To this end, we introduce the following sentence from $\text{dFO}^2[2, \Theta \cup \{\text{eq}, \text{ed}\}, \Gamma_{df}]$:

$$\begin{aligned} \xi_{\text{ed}}^\Theta := & \bigwedge_{i \in \{1, 2\}} \forall x. \exists y. (\text{ed}(y) \wedge x i \sim_i y) \wedge (\forall x. \forall y. (\text{ed}(x) \wedge \text{ed}(y) \wedge x i \sim_i y) \rightarrow x = y) \\ & \wedge \forall x. \text{eq}(x) \leftrightarrow \exists y. (\text{ed}(y) \wedge x 1 \sim_1 y \wedge x 2 \sim_2 y) \\ & \wedge \forall x. \text{ed}(x) \rightarrow \bigwedge_{\sigma \in \Theta} \neg \sigma(x) \end{aligned}$$

Informally, this formula ensures the following properties:

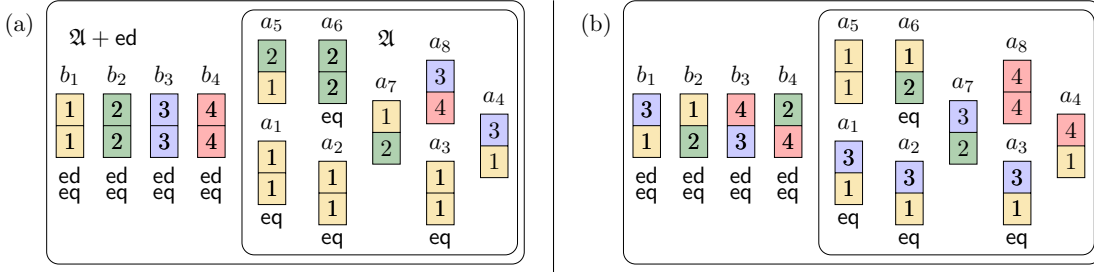


FIGURE 4. (a) Adding diagonal elements. (a) $\leftarrow$ (b) Making a data structure eq-respecting with the following permutation $1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 3$.

- for each element, and each of its value, there exists an element labeled by **ed** having the same value at the same position;
- there is no two distinct elements labeled by **ed** and having an equal value at the same position;
- for all elements labeled by **eq**, there exists an element labeled by **ed** having the two same values at the same positions;
- the elements labeled by **ed** are not labeled by any other labels from Θ .

Every structure that is well-diagonalized satisfies ξ_{ed}^Θ . The converse is not true in general. In particular, a model of ξ_{ed}^Θ is not necessarily eq-respecting. However, if a structure satisfies a formula $\varphi \in \text{dFO}[2, \Theta \cup \{\text{eq}, \text{ed}\}, \Gamma_{df}]$, then it is possible to perform a permutation on the first (or the second) values of its elements while preserving φ . This allows us to get:

Lemma 3. *Let $\mathfrak{B} \in \text{Data}[2, \Theta \cup \{\text{eq}, \text{ed}\}]$ and $\varphi \in \text{dFO}[2, \Theta \cup \{\text{eq}, \text{ed}\}, \Gamma_{df}]$. If $\mathfrak{B} \models \varphi \wedge \xi_{\text{ed}}^\Theta$, then there exists an eq-respecting $\mathfrak{A} \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$ such that $\mathfrak{A} + \text{ed} \models \varphi$.*

Proof. Let $\mathfrak{B} = (A, (P_\sigma), f_1, f_2)$ in $\text{Data}[2, \Theta \cup \{\text{eq}, \text{ed}\}]$ such that $\mathfrak{B} \models \varphi \wedge \xi_{\text{ed}}^\Theta$. We define the sets $I = \{n \in \mathbb{N} \mid \exists b \in P_{\text{ed}}. f_1(b) = n\}$ and $O = \{n \in \mathbb{N} \mid \exists b \in P_{\text{ed}}. f_2(b) = n\}$. Since

$$\mathfrak{B} \models \bigwedge_{i \in \{1, 2\}} [(\forall x. \exists y. \text{ed}(y) \wedge x \sim_i y) \wedge (\forall x. \forall y. (\text{ed}(x) \wedge \text{ed}(y) \wedge x \sim_i y) \rightarrow x = y)],$$

we deduce that $|I| = |O|$ and furthermore the mapping $\pi : I \mapsto O$ defined by $\pi(n) = m$ iff there exists $b \in P_{\text{ed}}$ such that $f_1(b) = n$ and $f_2(b) = m$ is a well-defined bijection. It is well defined because there is a single element b in P_{ed} such that $f_1(b) = n$ and it is a bijection because for all $m \in O$, there is a single $b \in P_{\text{ed}}$ such that $f_2(b) = m$. We can consequently extend π to be a permutation from $\mathbb{N}$ to $\mathbb{N}$. We then take the model $\mathfrak{A}' = (A, (P_\sigma), \pi \circ f_1, f_2)$. Since $\varphi \wedge \xi_{\text{ed}} \in \text{dFO}[2, \Theta \cup \{\text{eq}, \text{ed}\}, \Gamma_{df}]$ with $\Gamma_{df} = \{1 \sim_1, 2 \sim_2\}$ and since $\mathfrak{B} \models \varphi \wedge \xi_{\text{ed}}$, we deduce that $\mathfrak{A}' \models \varphi \wedge \xi_{\text{ed}}$ because performing a permutation on the first data values of the elements of $\mathfrak{B}$ does not affect the satisfaction of $\varphi \wedge \xi_{\text{ed}}$ (this is a consequence of the fact that there is no comparison between the first values and the second values of the elements). The satisfaction of ξ_{ed} by $\mathfrak{A}'$ allows us to deduce that $\mathfrak{A}'$ is well-diagonalized. We can in fact safely remove from $\mathfrak{A}'$ the elements of P_{ed} to obtain a structure $\mathfrak{A} \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$ which is eq-respecting (this is due to the fact that $\mathfrak{A}' \models \forall x. \text{eq}(x) \leftrightarrow \exists y. (\text{ed}(y) \wedge x \sim_1 y \wedge x \sim_2 y) \wedge \forall x. \text{ed}(x) \rightarrow \bigwedge_{\sigma \in \Theta \setminus \{\text{eq}\}} \neg \sigma(x)$) and such that $\mathfrak{A}' = \mathfrak{A} + \text{ed}$. $\square$

Example 8. Consider Figure 4 and let $\Theta = \emptyset$. The data structure from Figure 4(b) satisfies ξ_{ed}^Θ , though it is not well-diagonalized. Suppose it also satisfies $\varphi \in \text{dFO}[2, \{\text{eq}, \text{ed}\}, \Gamma_{df}]$. By permutation of the first data values ($1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 3$), we obtain the well-diagonalized data structure in Figure 4(a). As φ does not talk about the diagonal relation, satisfaction of φ is preserved.

Finally, we can inductively translate $\varphi \in \text{dFO}[2, \Theta \cup \{\text{eq}\}, \emptyset]$ into a formula $\llbracket \varphi \rrbracket_{+\text{ed}} \in \text{dFO}[2, \Theta \cup \{\text{eq}, \text{ed}\}, \emptyset]$ that avoids the extra ‘diagonal’ elements: $\llbracket \sigma(x) \rrbracket_{+\text{ed}} = \sigma(x)$, $\llbracket x = y \rrbracket_{+\text{ed}} = (x = y)$, $\llbracket \exists x. \varphi \rrbracket_{+\text{ed}} = \exists x. (\neg \text{ed}(x) \wedge \llbracket \varphi \rrbracket_{+\text{ed}})$, $\llbracket \varphi \vee \varphi' \rrbracket_{+\text{ed}} = \llbracket \varphi \rrbracket_{+\text{ed}} \vee \llbracket \varphi' \rrbracket_{+\text{ed}}$, and $\llbracket \neg \varphi \rrbracket_{+\text{ed}} = \neg \llbracket \varphi \rrbracket_{+\text{ed}}$. We immediately obtain:

Lemma 4. Let $\mathfrak{A} \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$ and $\varphi \in \text{dFO}[2, \Theta \cup \{\text{eq}\}, \emptyset]$ be a sentence. We have $\mathfrak{A} \models \varphi$ iff $\mathfrak{A} + \text{ed} \models \llbracket \varphi \rrbracket_{+\text{ed}}$.

Step 3: Getting Rid Of the Diagonal Relation

We will now exploit well-diagonalized data structures to reason about environments relative to Γ in terms of environments relative to Γ_{df} . Recall that Θ ranges over finite sets such that $\Sigma \subseteq \Theta$.

Lemma 5. Let $\mathfrak{A} = (A, \cdot, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$ be eq-respecting and $\mathfrak{B} = \mathfrak{A} + \text{ed}$. Moreover, let $a \in A$, $U \subseteq \Sigma$, and $R \subseteq \Gamma$ be a nonempty set. We have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) =$

$$\begin{cases} \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \Gamma_{df}) \setminus P_{\text{ed}} & \text{if } a \in P_{\text{eq}} \text{ and } R = \Gamma & (1) \\ \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \Gamma_{df}) & \text{if } a \notin P_{\text{eq}} \text{ and } R = \Gamma_{df} & (2) \\ \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim 1\}) \cap (P_{\text{eq}} \setminus P_{\text{ed}}) & \text{if } a \notin P_{\text{eq}} \text{ and } R = \{1 \sim 1, 1 \sim 2\} & (3) \\ \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{2 \sim 2\}) & \text{if } a \in P_{\text{eq}} \text{ and } R = \{2 \sim 2, 1 \sim 2\} & (4) \\ \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{2 \sim 2\}) \setminus P_{\text{ed}} & \text{if } a \notin P_{\text{eq}} \text{ and } R = \{2 \sim 2\} & (5) \\ \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim 1\}) \setminus P_{\text{eq}} & \text{if } R = \{1 \sim 1\} & (6) \\ \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(d, U, \{2 \sim 2\}) & \text{if } a \notin P_{\text{eq}} \text{ and } R = \{1 \sim 2\} & (7) \\ \text{for the unique } d \in P_{\text{ed}} \text{ such that } d \sim_1^{\mathfrak{B}} a & & \\ \emptyset & \text{otherwise} & (8) \end{cases}$$

Example 9. Before to provide the formal proof, let us go through the different cases of Lemma 5 using Figure 4(a), and letting $\Sigma = \Theta = \emptyset$.

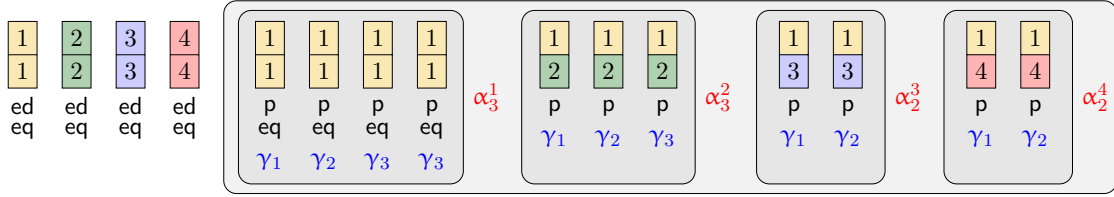
- (1) Let $a = a_1$ and $R = \Gamma$. Then, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \{a_1, a_2, a_3\}$. We also have that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \Gamma_{df}) = \{a_1, a_2, a_3, b_1\}$: These are the elements that coincide with a exactly on the first and the on the second data value when we dismiss the diagonal relation. Of course, as we consider $\mathfrak{B}$, this includes b_1 , which we have to exclude. Thus, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \Gamma_{df}) \setminus P_{\text{ed}}$.
- (2) Let $a = a_7$ and $R = \Gamma_{df}$. Then, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \Gamma_{df}) = \{a_7\}$. Since $a \notin P_{\text{eq}}$, it actually does not matter whether we include the diagonal relation or not.
- (3) Let $a = a_7$ and $R = \{1 \sim 1, 1 \sim 2\}$. Then, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \{a_1, a_2, a_3\}$. So how do we get this set in $\mathfrak{B}$ without referring to the diagonal relation? The idea is to use only $1 \sim 1 \in \Gamma_{df}$ and to ensure data equality by restricting to elements in P_{eq} (again excluding P_{ed}). Indeed, we have $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \{1 \sim 1\}) \cap (P_{\text{eq}} \setminus P_{\text{ed}}) = \{a_1, a_2, a_3, b_1\} \cap (\{a_1, a_2, a_3, b_1\} \setminus \{b_1\}) = \{a_1, a_2, a_3\}$.

- (4) Let $a = a_1$ and $R = \{2 \sim 2, 1 \sim 2\}$. Then, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \{a_4, a_5\}$. So we are looking for elements that have 1 as the second data value and a first data value different from 1, and this set is exactly $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \{2 \sim 2\})$.
- (5) Let $a = a_5$ and $R = \{2 \sim 2\}$. Then, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \{a_1, a_2, a_3, a_4\}$, which is the set of elements that have 1 as the second data value and a first data value different from 2. Thus, this is exactly $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \{2 \sim 2\}) \setminus P_{\text{ed}}$ (i.e., after discarding $b_1 \in P_{\text{ed}}$).
- (6) Let $a = a_4$ and $R = \{1 \sim 1\}$. We have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \{a_8\}$. Looking at $\mathfrak{B}$ and discarding the diagonal relation would also include b_3 and any element with data-value pair $(3, 3)$. Discarding P_{eq} , we obtain $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, \emptyset, \{1 \sim 1\}) \setminus P_{\text{eq}} = \{a_8, b_3\} \setminus \{b_3\} = \{a_8\}$.
- (7) Let $a = a_7$ and $R = \{1 \sim 2\}$. Then, $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, \emptyset, R) = \{a_4, a_5\}$, which is the set of elements whose second data value is 1 and whose first data value is different from 1. The idea is now to change the reference point. Take the unique $d \in P_{\text{ed}}$ such that $d \sim_1^{\mathfrak{B}} a$. Thus, $d = b_1$. The set $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(b_1, \emptyset, \{2 \sim 2\})$ gives us exactly the elements that have 1 as the second data value and a first value different from 1, as desired.

Proof. Let $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Theta \cup \{\text{eq}\}]$ be eq-respecting and $\mathfrak{B} = \mathfrak{A} + \text{ed}$. We consider $a \in A$, $U \subseteq \Sigma$, and $R \subseteq \Gamma$ be a nonempty set. Note that by definition of Env , we have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) \setminus P_{\text{ed}}$ and when $R \neq \{1 \sim 2\}$, we have $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, R \setminus \{(1, 2)\}) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R \cup \{1 \sim 2\}) \cup \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R \setminus \{1 \sim 2\})$. We will use these two equalities in the rest of the proof. We now perform a case analysis on R .

- (1) Assume $R = \Gamma = \{1 \sim 1, 2 \sim 2, 1 \sim 2\}$. First we suppose that $a \notin P_{\text{eq}}$. Since $\mathfrak{A}$ is eq-respecting it implies that $f_1(a) \neq f_2(a)$. Now assume there exists $b \in \text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R)$, this means that $a \sim_1^{\mathfrak{A}} b$ and $a \sim_2^{\mathfrak{A}} b$ and $a \sim_1^{\mathfrak{A}} b$. Hence we have $f_2(a) = f_2(b)$ and $f_1(a) = f_2(b)$, which is a contradiction. Consequently $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \emptyset$. We now suppose that $a \in P_{\text{eq}}$. In that case, since $\mathfrak{A}$ is eq-respecting, we have $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R \setminus \{1 \sim 2\}) = \emptyset$. In fact if $a \sim_2^{\mathfrak{A}} b$ for some b then $a \sim_1^{\mathfrak{A}} b$ as $a \in P_{\text{eq}}$. Hence we have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) \setminus P_{\text{ed}} = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \Gamma_{df}) \setminus P_{\text{ed}}$.
- (2) Assume $R = \Gamma_{df} = \{1 \sim 1, 2 \sim 2\}$. By a similar reasoning as the previous case, if $a \in P_{\text{eq}}$, we have necessarily $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \emptyset$. Now suppose $a \notin P_{\text{eq}}$. Thanks to this hypothesis, we know that $P_{\text{ed}} \cap \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \Gamma_{df}) = \emptyset$ and that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim 1, 2 \sim 2, (1, 2)\}) = \emptyset$. Hence we obtain directly $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, \Gamma_{df}) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \Gamma_{df})$.
- (3) Assume $R = \{1 \sim 1, 1 \sim 2\}$. Again it is obvious that if $a \in P_{\text{eq}}$, we have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \emptyset$. We suppose that $a \notin P_{\text{eq}}$. Note that we have that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) \subseteq P_{\text{eq}}$ and $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim 1\}) \cap P_{\text{eq}} = \emptyset$. Since $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim 1 s\}) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) \cup \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{(1, 1)\})$, we deduce that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{(1, 1)\}) \cap P_{\text{eq}} = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R)$. From which we can conclude $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim 1\}) \cap P_{\text{eq}} \setminus P_{\text{ed}}$.
- (4) Assume $R = \{2 \sim 2, 1 \sim 2 s\}$. Again, it is obvious that if $a \notin P_{\text{eq}}$, we have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \emptyset$. We now suppose that $a \in P_{\text{eq}}$. In that case, we have that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R \setminus \{1 \sim 2\}) = \emptyset$ and furthermore $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) \cap P_{\text{ed}} = \emptyset$. We can hence conclude that $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{2 \sim 2\})$.
- (5) Assume $R = \{2 \sim 2\}$. As before if $a \in P_{\text{eq}}$, we have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \emptyset$. We now suppose that $a \notin P_{\text{eq}}$. In that case, we have immediately $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{2 \sim 2, 1 \sim 2\}) = \emptyset$ and consequently

$$\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{2 \sim 2\}) \setminus P_{\text{ed}} = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{2 \sim 2\}) \setminus P_{\text{ed}}.$$

FIGURE 5. Counting intersections for $M = 3$ and elements with label p

- (6) Assume $R = \{1 \sim_1\}$. Remember that we have $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{(1, 1)\}) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim_1, 1 \sim_2\}) \cup \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim_1\})$. But $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim_1, 1 \sim_2\}) \subseteq P_{\text{eq}}$ and, moreover, $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim_1\}) \cap P_{\text{eq}} = \emptyset$. We hence deduce that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim_1\}) \setminus P_{\text{eq}} = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, \{1 \sim_1\})$ and since $(\text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim_1\}) \setminus P_{\text{eq}}) \cap P_{\text{ed}} = \emptyset$, we obtain $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(a, U, \{1 \sim_1\}) \setminus P_{\text{eq}}$.
- (7) Assume $R = \{1 \sim_2\}$. Again it is obvious that if $a \in P_{\text{eq}}$, we have $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R) = \emptyset$. We now suppose that $a \notin P_{\text{eq}}$. By definition, since $\mathfrak{B} = \mathfrak{A} + \text{ed}$, in $\mathfrak{B}$ there is a unique $d \in P_{\text{ed}}$ such that $d \sim_1^{\mathfrak{B}} a$. We have then $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(d, U, \{1 \sim_2, 2 \sim_2\})$. As for the case 4., we deduce that $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(d, U, \{1 \sim_2, 2 \sim_2\}) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(d, U, \{2 \sim_2\})$. Hence $\text{Env}_{\mathfrak{B}, \Sigma, \Gamma}(a, U, R) = \text{Env}_{\mathfrak{B}, \Sigma, \Gamma_{df}}(d, U, \{2 \sim_2\})$. $\square$

Let us wrap up: By Lemmas 2 and 5, we end up with checking counting constraints in an extended data structure without using the diagonal relation.

Step 4: Counting in Two-Variable Logic

The next step is to express these constraints using two-variable formulas. Counting in two-variable logic is established using further unary predicates. These additional predicates allow us to define a partitioning of the universe of a structure into so-called *intersections*. Suppose $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Theta \cup \{\text{eq}, \text{ed}\}]$, where $\Sigma \subseteq \Theta$. Let $a \in A \setminus P_{\text{ed}}$ and define $\ell_\Sigma(a) = \{\sigma \in \Sigma \mid a \in P_\sigma\}$. The *intersection* of a in $\mathfrak{A}$ is the set $\{b \in A \setminus P_{\text{ed}} \mid a \sim_1 b \wedge a \sim_2 b \wedge \ell_\Sigma(a) = \ell_\Sigma(b)\}$. A set is called an *intersection* in $\mathfrak{A}$ if it is the intersection of some $a \in A \setminus P_{\text{ed}}$.

Example 10. Consider Figure 5 and suppose $\Sigma = \{p\}$. The intersections of the given data structure are gray-shaded.

Let us introduce the various unary predicates, which will be assigned to *non-diagonal* elements. There are three types of them (for the first two types, also see Figure 5):

- (1) The unary predicates $\Lambda_M^\gamma = \{\gamma_1, \dots, \gamma_M\}$ have the following intended meaning: For all intersections I and $i \in \{1, \dots, M\}$, we have $|I| \geq i$ iff there is $a \in I$ such that $a \in P_{\gamma_i}$. In other words, the presence (or absence) of γ_i in an intersection I tells us whether $|I| \geq i$.
- (2) The predicates $\Lambda_M^\alpha = \{\alpha_i^j \mid i \in \{1, \dots, M\} \text{ and } j \in \{1, \dots, M+2\}\}$ have the following meaning: If a is labeled with α_i^j , then (i) there are at least j intersections sharing the same first value and the same label set $\ell_\Sigma(a)$, and (ii) the intersection of a has i elements if $i \leq M-1$ and at least M elements if $i = M$. Hence, in α_i^j , index i counts the elements inside an intersection, and j labels up to $M+2$ different intersections. We need to

- go beyond M due to Lemma 5: When we remove certain elements (e.g., P_{eq}) from an environment, we must be sure to still have sufficiently many to be able to count until M .
- (3) Labels from $\Lambda_M^\beta = \{\beta_i^j \mid i \in \{1, \dots, M\} \text{ and } j \in \{1, \dots, M+1\}\}$ will play a similar role as those in Λ_M^α but consider the second values of the elements instead of the first ones.

Example 11. A suitable labeling for types γ and α is illustrated in Figure 5 for $M = 3$.

Let $\Lambda_M = \Lambda_M^\alpha \cup \Lambda_M^\beta \cup \Lambda_M^\gamma$ denote the set of all these unary predicates. We shall now see how to build sentences $\varphi_\alpha, \varphi_\beta, \varphi_\gamma \in \text{dFO}^2[2, \Theta \cup \{\text{eq}, \text{ed}\} \cup \Lambda_M, \Gamma_{df}]$ that guarantee the respective properties.

To deal with the predicates in Λ_M^γ , we first define the formula $\varphi_{\text{same}}^{\text{int}} = x \sim_1 y \wedge x \sim_2 y \wedge \bigwedge_{\sigma \in \Sigma} \sigma(x) \leftrightarrow \sigma(y)$ and introduce the following formulas:

$$\begin{aligned} \varphi_\gamma^1(x) &:= \bigvee_{i \in [1, M]} (\gamma_i(x) \wedge \bigwedge_{j \in [1, M] \setminus \{i\}} \neg \gamma_j(x)) \\ \varphi_\gamma^2(x) &:= \bigwedge_{i \in [1, M-1]} (\gamma_i(x) \rightarrow \neg \exists y. (x \neq y \wedge \varphi_{\text{same}}^{\text{int}}(x, y) \wedge \gamma_i(y))) \\ \varphi_\gamma^3(x) &:= \bigwedge_{i \in [2, M]} (\gamma_i(x) \rightarrow (\exists y. \varphi_{\text{same}}^{\text{int}}(x, y) \wedge \gamma_{i-1}(y))) \end{aligned}$$

We then let $\varphi_\gamma := \forall x. (\neg \text{ed}(x) \rightarrow (\varphi_\gamma^1(x) \wedge \varphi_\gamma^2(x) \wedge \varphi_\gamma^3(x))) \wedge (\text{ed}(x) \rightarrow \bigwedge_{\gamma \in \Lambda_M^\gamma} \neg \gamma(x))$. Thus, a data structure satisfies φ_γ if no diagonal element is labelled with predicates in Λ_M^γ and (1) all its non-diagonal elements are labelled with exactly one predicate in Λ_M^γ (see φ_γ^1), (2) if $i \leq M-1$, then there are no two γ_i -labelled elements with the same labels of Σ and in the same intersection (see φ_γ^2), and (3) if $i \geq 2$, then for all γ_i -labelled elements, there exists an γ_{i-1} -labelled element with the same labels of Σ and in the same intersection (see φ_γ^3).

Lemma 6. Let $\mathfrak{A} = (A, (P_\sigma), f_1, f_2,) \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathbb{C}_M \cup \Lambda_M]$ be eq-respecting and such that $\mathfrak{A} + \text{ed} \models \varphi_\gamma$. We consider $a \in A$ and $\gamma_i \in \Lambda_M$ and $E_a = \{b \in A \mid a \sim_1 b \wedge a \sim_2 b \wedge b \wedge \ell_\Sigma(a) = \ell_\Sigma(b)\}$. Then, $|E_a| \geq i$ iff there exists $b \in E_a$ such that $b \in P_{\gamma[i]}$.

Proof. For any $b \in E_a$, as $\mathfrak{A} + \text{ed} \models \varphi_\gamma^1$ there is exactly one $j \in [1, M]$ such that $b \in P_{\gamma_j}$. This allow us to build the function $f : E_a \rightarrow [1, M]$ which associates to any $b \in E_a$ such a j . Let $J = \{f(b) \mid b \in E_a\}$ denotes the image of E_a under f . As $\mathfrak{A} + \text{ed} \models \varphi_\gamma^3$, for any $j \in [2, M]$ if $j \in J$ then $j-1 \in J$. And as $E_a \neq \emptyset$, there is $j_{\max} \in [1, M]$ such that $J = [1, j_{\max}]$. We can now rephrase our goal as $|E_a| \geq i$ iff $i \in J$. Assuming that $i \in J$, we have $i \leq j_{\max}$. As f is a function, we have $|E_a| \geq |J|$. As $|J| = j_{\max}$, we have that $|E_a| \geq i$. Conversely, assuming that $|E_a| \geq i$. Assume by contradiction that $i \notin J$, then $j_{\max} < i \leq M$. That is, for all $j \in J$, we have $j < M$. Since $\mathfrak{A} + \text{ed} \models \varphi_\gamma^2$, all elements of J have exactly one preimage. So $|E_a| = |J| = j_{\max} < i$, which contradicts the assumption. $\square$

It is then easy to see that, in an intersection, if there is an element a labelled by γ_i and no element labelled by γ_{i+1} for $i < M$, then the intersection has exactly i elements; moreover, if there is a node a labelled by γ_M then the intersection has at least M elements.

We now show how we use the predicates in Λ_M^α and introduce the following formulas (where $\varphi_{same}^{int} = x \sim_1 y \wedge x \sim_2 y \wedge \bigwedge_{\sigma \in \Sigma} \sigma(x) \leftrightarrow \sigma(y)$ and $\varphi_{same} = \bigwedge_{\sigma \in \Sigma} \sigma(x) \leftrightarrow \sigma(y)$):

$$\begin{aligned}
\varphi_\alpha^1(x) &:= \bigvee_{\substack{i \in [1, M] \\ j \in [1, M+2]}} \left(\alpha_i^j(x) \wedge \bigwedge_{\substack{k \in [1, M] \\ \ell \in [1, M+2] \\ (k, \ell) \neq (i, j)}} \neg \alpha_k^\ell(x) \right) \\
\varphi_\alpha^2(x) &:= \bigwedge_{\substack{i \in [1, M] \\ j \in [1, M+2]}} \left(\alpha_i^j(x) \rightarrow \forall y. ((\neg \text{ed}(y) \wedge \varphi_{same}^{int}(x, y)) \rightarrow \alpha_i^j(y)) \right) \\
\varphi_\alpha^3(x) &:= \bigwedge_{\substack{i \in [1, M-1] \\ j \in [1, M+2]}} \left(\alpha_i^j(x) \rightarrow \left(\begin{array}{c} \exists y. (\varphi_{same}^{int}(x, y) \wedge \gamma_i(y)) \\ \wedge \neg \exists y. (\varphi_{same}^{int}(x, y) \wedge \gamma_{i+1}(y)) \end{array} \right) \right) \wedge \\
&\quad \bigwedge_{j \in [1, M+2]} \left(\alpha_M^j(x) \rightarrow \exists y. (\varphi_{same}^{int}(x, y) \wedge \gamma_M(y)) \right) \\
\varphi_\alpha^4(x) &:= \bigwedge_{\substack{i \in [1, M] \\ j \in [1, M+1]}} \left(\alpha_i^j(x) \rightarrow \forall y. \left(\left(\begin{array}{c} \neg \text{ed}(y) \wedge \varphi_{same}(x, y) \\ \wedge x \sim_1 y \wedge \neg(x \sim_2 y) \end{array} \right) \rightarrow \bigwedge_{k \in [1, M]} \neg \alpha_k^j(y) \right) \right) \\
\varphi_\alpha^5(x) &:= \bigwedge_{\substack{i \in [1, M] \\ j \in [2, M+2]}} \left(\alpha_i^j(x) \rightarrow \exists y. (\varphi_{same}(x, y) \wedge x \sim_1 y \wedge \bigvee_{k \in [1, M]} \alpha_k^{j-1}(y)) \right)
\end{aligned}$$

We then define $\varphi_\alpha := \forall x. ((\neg \text{ed}(x)) \rightarrow (\varphi_\alpha^1(x) \wedge \varphi_\alpha^2(x) \wedge \varphi_\alpha^3(x) \wedge \varphi_\alpha^4(x) \wedge \varphi_\alpha^5(x))) \wedge (\text{ed}(x) \rightarrow \bigwedge_{\alpha \in \Lambda_M^\alpha} \neg \alpha(x))$. Note that φ_α is a two-variable formula in $\text{dFO}^2[2, \Theta \cup \{\text{ed}\} \cup \Lambda_M, \Gamma_{df}]$. If a data structure satisfies φ_α , then no diagonal element is labelled with predicates in Λ_M^α and all its non-diagonal elements are labelled with exactly one predicate in Λ_M^α (see φ_α^1). Furthermore, all non-diagonal elements in a same intersection are labelled with the same α_i^j (see φ_α^2), and there are exactly i such elements in the intersection if $i \leq M-1$ and at least M otherwise (see φ_α^3). Finally, we want to identify up to $M+2$ different intersections sharing the same first value and we use the j in α_i^j for this matter. Formula φ_α^4 tells us that no two non-diagonal elements with the same labels of Σ share the same index j (for $j \leq M+1$) if they do not belong to the same intersection and have the same first value. The formula φ_α^5 specifies that, if an element a is labelled with α_i^j , then there are at least j different nonempty intersections with the same labels of Σ as a sharing the same first values.

The next lemma formalizes the property of this labelling.

Lemma 7. *We consider $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathbb{C}_M \cup \Lambda_M]$ eq-respecting and such that $\mathfrak{A} + \text{ed} \models \varphi_\gamma \wedge \varphi_\alpha$ and $a \in A$. Let $S_{a,1 \sim 1} = \{b \in A \mid a \sim_1^{\mathfrak{A}} b \wedge \ell_\Sigma(a) = \ell_\Sigma(b)\}$ and $S_{a,1 \sim 1, i}^j = S_{a,1 \sim 1} \cap P_{\alpha_i^j}$ for all $i \in [1, M]$ and $j \in [1, M+2]$. The following properties hold:*

- (1) *We have $S_{a,1 \sim 1} = \bigcup_{i \in [1, M], j \in [1, M+2]} S_{a,1 \sim 1, i}^j$.*

- (2) For all $j, \ell \in [1, M+2]$ and $i, k \in [1, M]$ such that $i \neq k$ or $j \neq \ell$, we have $S_{a,1 \sim_1, i}^j \cap S_{a,1 \sim_1, k}^\ell = \emptyset$.
- (3) For all $j \in [1, M+1]$ and $i \in [1, M]$ such that $b, c \in S_{a,1 \sim_1, i}^j$, we have $b \sim_2 c$.
- (4) For all $b, c \in S_{a,1 \sim_1}$ such that $b \sim_2 c$, there exist $j \in [1, M+2]$ and $i \in [1, M]$ such that $b, c \in S_{a,1 \sim_1, i}^j$.
- (5) For all $j \in [1, M+2]$ and $i \in [1, M]$ such that $b \in S_{a,1 \sim_1, i}^j$, we have

$$\begin{cases} |\{c \in A \mid b \sim_1^{\mathfrak{A}} c \wedge b \sim_2^{\mathfrak{A}} c \wedge \ell_\Sigma(b) = \ell_\Sigma(c)\}| = i & \text{if } i \leq M-1 \\ |\{c \in A \mid b \sim_1^{\mathfrak{A}} c \wedge b \sim_2^{\mathfrak{A}} c \wedge \ell_\Sigma(b) = \ell_\Sigma(c)\}| \geq M & \text{otherwise.} \end{cases}$$

- (6) For all $j \in [1, M+1]$, there exists at most one i such that $S_{a,1 \sim_1}^{j, [i]} \neq \emptyset$.
- (7) For all $j \in [2, M+2]$ and $i \in [1, M]$ such that $S_{a,1 \sim_1, i}^j \neq \emptyset$, there exists $k \in [1, M]$ such that $S_{a,1 \sim_1, k}^{j-1} \neq \emptyset$.

Proof. We prove the different statements:

- (1) Thanks to the formula $\varphi_\alpha^1(x)$ we have that $A = \bigcup_{i \in [1, M], j \in [1, M+2]} P_{\alpha_i^j}$. Since $S_{a,1 \sim_1} = A \cap S_{a,1 \sim_1}$, we deduce that

$$S_{a,1 \sim_1} = \left(\bigcup_{i \in [1, M], j \in [1, M+2]} P_{\alpha_i^j} \right) \cap S_{a,1 \sim_1} = \bigcup_{i \in [1, M], j \in [1, M+2]} S_{a,1 \sim_1, i}^j.$$

- (2) This point can be directly deduced thanks to $\varphi_\alpha^1(x)$.
- (3) This point can be directly deduced thanks to $\varphi_\alpha^4(x)$.
- (4) Since $b \in S_{a,1 \sim_1}$, by 1. there exist $j \in [1, M+2]$ and $i \in [1, M]$ such that $b \in S_{a,1 \sim_1, i}^j$. Furthermore, since $c \in S_{a,1 \sim_1}$, using formula $\varphi_\alpha^2(x)$, we deduce that $c \in S_{a,1 \sim_1, i}^j$.
- (5) This point can be directly deduced thanks to formula $\varphi_\alpha^3(x)$ and to Lemma 6.
- (6) Assume there exist $i, i' \in [1, M]$ such that $i \neq i'$ and $S_{a,1 \sim_1, i}^j \neq \emptyset$ and $S_{a,1 \sim_1, i'}^j \neq \emptyset$. Let $b \in S_{a,1 \sim_1, i}^j$ and $c \in S_{a,1 \sim_1, i'}^j \neq \emptyset$. If $b \sim_2^{\mathfrak{A}} c$, then, by 5., we necessarily have $i = i'$. Hence we deduce that $b \sim_2^{\mathfrak{A}} c$ does not hold, and we can conclude thanks to formula $\varphi_\alpha^4(x)$.
- (7) This point can be directly deduced thanks to formula $\varphi_\alpha^5(x)$. $\square$

While the predicates α_i^j deal with the relation $1 \sim_1$, we now define a similar formula $\varphi_\beta \in \text{dFO}^2[2, \Theta \cup \{\text{ed}\} \cup \Lambda_M, \Gamma_{df}]$ for the predicates in Λ_M^β to count intersections connected by the binary relation $2 \sim_2$.

Remark 1. The formula φ_β to deal with the predicates in Λ_M^β is built the exact same way as the formula φ_α but we provide its definition and its properties in a detailed way for the sake of completeness. The already convinced reader can continue right after Lemma 8.

We introduce hence the following formulas (we recall that $\varphi_{\text{same}}^{\text{int}} = x \sim_1 y \wedge x \sim_2 y \wedge \bigwedge_{\sigma \in \Sigma} \sigma(x) \leftrightarrow \sigma(y)$ and $\varphi_{\text{same}} = \bigwedge_{\sigma \in \Sigma} \sigma(x) \leftrightarrow \sigma(y)$):

$$\varphi_{\beta}^1(x) := \bigvee_{\substack{i \in [1, M] \\ j \in [1, M+1]}} \left(\beta_i^j(x) \wedge \bigwedge_{\substack{k \in [1, M] \\ \ell \in [1, M+1] \\ (k, \ell) \neq (i, j)}} \neg \beta_k^\ell(x) \right)$$

$$\varphi_{\beta}^2(x) := \bigwedge_{\substack{i \in [1, M] \\ j \in [1, M+1]}} \left(\beta_i^j(x) \rightarrow \forall y. ((\neg \text{ed}(y) \wedge \varphi_{\text{same}}^{\text{int}}(x, y)) \rightarrow \beta_i^j(y)) \right)$$

$$\varphi_{\beta}^3(x) := \bigwedge_{\substack{i \in [1, M-1] \\ j \in [1, M+1]}} \left(\beta_i^j(x) \rightarrow \left(\begin{array}{c} \exists y. (\varphi_{\text{same}}^{\text{int}}(x, y) \wedge \gamma_i(y)) \\ \wedge \neg \exists y. (\varphi_{\text{same}}^{\text{int}}(x, y) \wedge \gamma_{i+1}(y)) \end{array} \right) \right) \wedge \\ \bigwedge_{j \in [1, M+1]} \left(\beta_M^j(x) \rightarrow \exists y. (\varphi_{\text{same}}^{\text{int}}(x, y) \wedge \gamma_M(y)) \right)$$

$$\varphi_{\beta}^4(x) := \bigwedge_{\substack{i \in [1, M] \\ j \in [1, M]}} \left(\beta_i^j(x) \rightarrow \forall y. \left(\begin{array}{c} \neg \text{ed}(y) \wedge \varphi_{\text{same}}(x, y) \\ \wedge \neg (x \sim_1 y) \wedge x \sim_2 y \end{array} \right) \rightarrow \bigwedge_{k \in [1, M]} \neg \beta_k^j(y) \right)$$

$$\varphi_{\beta}^5(x) := \bigwedge_{\substack{i \in [1, M] \\ j \in [2, M+1]}} \left(\beta_i^j(x) \rightarrow \exists y. \left(\varphi_{\text{same}}(x, y) \wedge x \sim_2 y \wedge \bigvee_{k \in [1, M+1]} \beta_k^{j-1}(y) \right) \right)$$

We then define $\varphi_{\beta} := \forall x. ((\neg \text{ed}(x)) \rightarrow (\varphi_{\beta}^1(x) \wedge \varphi_{\beta}^2(x) \wedge \varphi_{\beta}^3(x) \wedge \varphi_{\beta}^4(x))) \wedge (\text{ed}(x) \rightarrow \bigwedge_{\beta \in \Lambda_M^{\infty}} \neg \beta(x))$.

The following Lemma is the equivalent of the Lemma 7 for the relation $\sim_2$. Its proof is similar to the one of the Lemma 7.

Lemma 8. *We consider $\mathfrak{A} = (A, (P_{\sigma}), f_1, f_2) \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathbb{C}_M \cup \Lambda_M]$ eq-respecting and such that $\mathfrak{A} + \text{ed} \models \varphi_{\gamma} \wedge \varphi_{\beta}$ and $a \in A$. Let $S_{a, 2 \sim_2} = \{b \in A \mid a \sim_2^{\mathfrak{A}} b \wedge \ell_{\Sigma}(a) = \ell_{\Sigma}(b)\}$ and $S_{a, 2 \sim_2, i}^j = S_{a, 2 \sim_2} \cap P_{\beta_i^j}$ for all $i \in [1, M]$ and $j \in [1, M+1]$. The following statements hold:*

- (1) *We have $S_{a, 2 \sim_2} = \bigcup_{i \in [1, M], j \in [1, M+1]} S_{a, 2 \sim_2, i}^j$.*
- (2) *For all $j, \ell \in [1, M+1]$ and $i, k \in [1, M]$ such that $i \neq k$ or $j \neq \ell$, we have $S_{a, 2 \sim_2, i}^j \cap S_{a, 2 \sim_2, k}^{\ell} = \emptyset$.*
- (3) *For all $j \in [1, M+1]$ and $i \in [1, M]$ such that $b, c \in S_{a, 2 \sim_2, i}^j$, we have $b \sim_1 c$.*
- (4) *For all $b, c \in S_{a, 2 \sim_2}$ such that $b \sim_1 c$, there exists $j \in [1, M+1]$ and $i \in [1, M]$ such that $b, c \in S_{a, 2 \sim_2, i}^j$.*
- (5) *For all $j \in [1, M+1]$ and $i \in [1, M]$ such that $b \in S_{a, 2 \sim_2, i}^j$, we have*

$$\begin{cases} |\{c \in A \mid b \sim_1^{\mathfrak{A}} c \wedge b \sim_2^{\mathfrak{A}} c \wedge \ell_{\Sigma}(b) = \ell_{\Sigma}(c)\}| = i & \text{if } i \leq M-1 \\ |\{c \in A \mid b \sim_1^{\mathfrak{A}} c \wedge b \sim_2^{\mathfrak{A}} c \wedge \ell_{\Sigma}(b) = \ell_{\Sigma}(c)\}| \geq M & \text{otherwise.} \end{cases}$$

- (6) For all $j \in [1, M]$, there exists at most one i such that $S_{a,2 \sim 2,i}^j \neq \emptyset$.
- (7) For all $j \in [2, M+1]$ and $i \in [1, M]$ such that $S_{a,2 \sim 2,i}^j \neq \emptyset$, there exists $k \in [1, M]$ such $S_{a,2 \sim 2,k}^{j-1} \neq \emptyset$.

Now that we can count on a consistent labeling with predicates from Λ_M , let us see how we can exploit it to express $\langle U, R, m \rangle \in \mathbb{C}_M$, with additional help from Lemma 5, as a formula $\varphi_{U,R,m}(x) \in \text{dFO}^2[2, \Theta \cup \{\text{eq}, \text{ed}\} \cup \Lambda_M, \Gamma_{df}]$ applied to *non-diagonal* elements (outside P_{ed}). Let us look at two sample cases according to the case distinction done in Lemma 5. Hereby, we will use, for $U \subseteq \Sigma$, the formula $\varphi_U(y) = \bigwedge_{\sigma \in U} \sigma(y) \wedge \bigwedge_{\sigma \in \Sigma \setminus U} \neg \sigma(y)$. We now provide the definition of the formulas $\varphi_{U,R,m}$ using a case analysis on the shape of R and the result of Lemma 5:

- (1) **Case** $R = \{1 \sim 1, 2 \sim 2, 1 \sim 2\}$: in this simple case, we need to say that (i) the element a under consideration is in P_{eq} , and (ii) there is an intersection of size at least m (i.e., it contains a γ_m -labeled element) whose elements b satisfy $a \sim_1 b$, $a \sim_2 b$, and $\ell_\Sigma(b) = U$:

$$\varphi_{U,R,m}(x) := \text{eq}(x) \wedge \exists y. (\varphi_U(y) \wedge x \sim_1 y \wedge x \sim_2 y \wedge \gamma_m(y))$$

- (2) **Case** $R = \{1 \sim 1, 2 \sim 2\}$:

$$\varphi_{U,R,m}(x) := \neg \text{eq}(x) \wedge \exists y. (\varphi_U(y) \wedge x \sim_1 y \wedge x \sim_2 y \wedge \gamma_m(y))$$

- (3) **Case** $R = \{1 \sim 1, 1 \sim 2\}$:

$$\varphi_{U,R,m}(x) := \neg \text{eq}(x) \wedge \exists y. (\varphi_U(y) \wedge \text{eq}(y) \wedge x \sim_1 y \wedge \gamma_m(y))$$

- (4) **Case** $R = \{2 \sim 2, 1 \sim 2\}$: For this case, we first need an extra definition. For $m \in [1, M]$, we define $\mathcal{S}_{\beta,m}$ the set of subsets of Λ_M^α as follows: $\mathcal{S}_{\beta,m} = \{\{\beta_{i_1}^{j_1}, \dots, \beta_{i_k}^{j_k}\} \mid i_1 + \dots + i_k \geq m \text{ and } j_1 < j_2 < \dots < j_k\}$. It corresponds to the sets of element β_i^j whose sum of i is greater than or equal to m . We then have:

$$\varphi_{U,R,m}(x) := \text{eq}(x) \wedge \bigvee_{S \in \mathcal{S}_{\beta,m}} \bigwedge_{\beta \in S} \exists y. (\varphi_U(y) \wedge \neg \text{eq}(y) \wedge \beta(y) \wedge x \sim_2 y)$$

- (5) **Case** $R = \{2 \sim 2\}$: we use again the set $\mathcal{S}_{\beta,m}$ introduced previously.

$$\varphi_{U,R,m}(x) := \neg \text{eq}(x) \wedge \bigvee_{S \in \mathcal{S}_{\beta,m}} \bigwedge_{\beta \in S} \exists y. (\varphi_U(y) \wedge \beta(y) \wedge \neg(x \sim_1 y) \wedge x \sim_2 y)$$

- (6) **Case** $R = \{1 \sim 1\}$: Similar to Case 4., we first need an extra definition. For $m \in \{1, \dots, M\}$, we define the set $\mathcal{S}_{\alpha,m}$ of subsets of Λ_M^α as follows: $\mathcal{S}_{\alpha,m} = \{\{\alpha_{i_1}^{j_1}, \dots, \alpha_{i_k}^{j_k}\} \mid i_1 + \dots + i_k \geq m \text{ and } j_1 < j_2 < \dots < j_k\}$. It corresponds to the sets of elements α_i^j whose sum of i is greater than or equal to m . We then have:

$$\varphi_{U,R,m}(x) := \bigvee_{S \in \mathcal{S}_{\alpha,m}} \bigwedge_{\alpha \in S} \exists y. (\varphi_U(y) \wedge \alpha(y) \wedge \neg \text{eq}(y) \wedge x \sim_1 y \wedge \neg(x \sim_2 y))$$

- (7) **Case** $R = \{1 \sim 2\}$: We use here again the set $\mathcal{S}_{\beta,m}$ introduced in Case 4.

$$\begin{aligned} \varphi_{U,R,m}(x) := & \neg \text{eq}(x) \wedge \exists y. \left(\text{ed}(y) \wedge x \sim_1 y \wedge \right. \\ & \left. \bigvee_{S \in \mathcal{S}_{\beta,m}} \bigwedge_{\sigma \in S} \exists x. (\varphi_U(x) \wedge \sigma(x) \wedge \neg(y \sim_1 x) \wedge y \sim_2 x) \right) \end{aligned}$$

Finally, it remains to say that all elements are labeled with the suitable counting constraints. So we let $\varphi_{cc} = \forall x. \neg \text{ed}(x) \rightarrow \bigwedge_{\langle U, R, m \rangle \in \mathcal{C}_M} \langle U, R, m \rangle(x) \leftrightarrow \varphi_{U, R, m}(x)$.

Lemma 9. *Let $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M \cup \Lambda_M]$ be eq-respecting. If $\mathfrak{A} + \text{ed} \models \varphi_\alpha \wedge \varphi_\beta \wedge \varphi_\gamma \wedge \varphi_{cc}$, then $\mathfrak{A}$ is cc-respecting.*

Proof. Let $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M \cup \Lambda_M]$ be eq-respecting and such that $\mathfrak{A} + \text{ed} \models \varphi_\alpha \wedge \varphi_\beta \wedge \varphi_\gamma \wedge \varphi_{cc}$. We need to show that for all $a \in A$ and all $\langle U, R, m \rangle \in \mathcal{C}_M$, we have $a \in P_{\langle U, R, m \rangle}$ iff $|\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R)| \geq m$. We consider $a \in A$. Since $\mathfrak{A} + \text{ed} \models \varphi_{cc}$, we deduce that $a \in P_{\langle U, R, m \rangle}$ iff $\mathfrak{A} + \text{ed} \models_{I[x/a]} \varphi_{U, R, m}(x)$. We need hence to show that $\mathfrak{A} + \text{ed} \models_{I[x/a]} \varphi_{U, R, m}(x)$ iff $|\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R)| \geq m$. To prove this, we first use Lemma 5 to get a characterization of $\text{Env}_{\mathfrak{A}, \Sigma, \Gamma}(a, U, R)$. This characterization is then directly translated into the formula $\varphi_{U, R, m}(x)$ which makes use of the label in Λ_M to count in the environment of a . The fact that this counting is performed correctly is guaranteed by Lemmas 6, 7 and 8. Putting these arguments together, we can conclude that the lemma holds. $\square$

Step 5: Putting it All Together

Let $\text{All} = \Sigma \cup \{\text{eq}, \text{ed}\} \cup \mathcal{C}_M \cup \Lambda_M$ denote the set of all the unary predicates that we have introduced so far. Recall that, after Step 1, we were left with $M \geq 1$ and a formula $\varphi \in \text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M, \emptyset]$. The question is now whether φ has a well-typed model (i.e., a model that is eq-respecting and cc-respecting). Altogether, we get the following reduction:

Proposition 2. *Let $\varphi \in \text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M, \emptyset]$. Then, φ has a well-typed model iff $\widehat{\varphi} := \llbracket \varphi \rrbracket_{+\text{ed}} \wedge \xi_{\text{ed}}^{\text{All} \setminus \{\text{eq}, \text{ed}\}} \wedge \varphi_\alpha \wedge \varphi_\beta \wedge \varphi_\gamma \wedge \varphi_{cc} \in \text{ext-dFO}^2[2, \text{All}, \Gamma_{df}]$ is satisfiable.*

Proof. Suppose $\widehat{\varphi}$ is satisfiable. Then, there is $\mathfrak{B} \in \text{Data}[2, \text{All}]$ such that $\mathfrak{B} \models \widehat{\varphi}$. By Lemma 3, there exists an eq-respecting data structure $\mathfrak{A} \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M \cup \Lambda_M]$ such that $\mathfrak{A} + \text{ed} \models \llbracket \varphi \rrbracket_{+\text{ed}} \wedge \varphi_\alpha \wedge \varphi_\beta \wedge \varphi_\gamma \wedge \varphi_{cc}$. Using Lemma 9, we deduce that $\mathfrak{A}$ is cc-respecting and, thus, well-typed. Furthermore, by Lemma 4, we have $\mathfrak{A} \models \varphi$. Note that $\mathfrak{A}$ belongs to $\text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M \cup \Lambda_M]$. However, by removing the unary predicates in Λ_M , we still have a model of φ from $\text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M]$ as required. Hence, φ has a well-typed model.

Assume now that there exists a well-typed data structure $\mathfrak{A} \in \text{Data}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M]$ such that $\mathfrak{A} \models \varphi$. Using Lemma 4, we have that $\mathfrak{A} + \text{ed} \models \llbracket \varphi \rrbracket_{+\text{ed}}$. Furthermore, using the fact that $\mathfrak{A}$ is well-typed, we can add the unary predicates from Λ_M to $\mathfrak{A} + \text{ed}$ to obtain a data structure $\mathfrak{A}' \in \text{Data}[2, \text{All}]$ such that $\mathfrak{A}' \models \varphi_\alpha \wedge \varphi_\beta \wedge \varphi_\gamma \wedge \varphi_{cc}$. Note that $\mathfrak{A}'$ is well-diagonalized. We deduce that $\mathfrak{A}' \models \widehat{\varphi}$. $\square$

We are now in the position to prove our main result:

Theorem 5. $\text{DATASAT}(1\text{-Loc-dFO}, 2, \{1 \sim 1, 2 \sim 2, 1 \sim 2\})$ is decidable.

Proof. Let $\psi \in 1\text{-Loc-dFO}[2, \Sigma, \{1 \sim 1, 2 \sim 2, 1 \sim 2\}]$. Using Lemma 2, we can effectively compute $M \in \mathbb{N}$ and $\varphi \in \text{dFO}[2, \Sigma \cup \{\text{eq}\} \cup \mathcal{C}_M, \emptyset]$ such that ψ is satisfiable iff φ has a well-typed model. By Proposition 2, φ has a well-typed model iff $\widehat{\varphi}$ is satisfiable. Since $\widehat{\varphi}$ belongs to $\text{ext-dFO}^2[2, \text{All}, \Gamma_{df}]$, we conclude using Proposition 1. $\square$

3.2. Undecidability results.

We shall now show that extending the neighborhood radius yields undecidability. We rely on a reduction from the domino problem [12] and use a specific technique presented in [31].

3.2.1. The Tiling Problem. A *domino system* $\mathcal{D}$ is a triple (D, H, V) where D is a finite set of dominoes and $H, V \subseteq D \times D$ are two binary relations. Let $\mathfrak{G}_m$ denote the standard grid on an $m \times m$ torus, i.e., $\mathfrak{G}_m = (G_m, H_m, V_m)$ where H_m and V_m are two binary relations defined as follows: $G_m = \mathbb{Z} \bmod m \times \mathbb{Z} \bmod m$, $H_m = \{((i, j), (i', j)) \mid i' - i \equiv 1 \bmod m\}$, and $V_m = \{((i, j), (i, j')) \mid j' - j \equiv 1 \bmod m\}$. In the sequel, we will suppose $\mathbb{Z} \bmod m = \{0, \dots, m-1\}$ using the least positive member to represent residue classes.

A *bi-binary structure* is a triple (A, R_1, R_2) where A is a finite set and R_1, R_2 are subsets of $A \times A$. Domino systems and $\mathfrak{G}_m$ for any m are examples of bi-binary structures. For two bi-binary structures $\mathfrak{G} = (G, H, V)$ and $\mathfrak{G}' = (G', H', V')$, we say that $\mathfrak{G}$ is *homomorphically embeddable* into $\mathfrak{G}'$ if there is a morphism $\pi : \mathfrak{G} \rightarrow \mathfrak{G}'$, i.e., a mapping π such that, for all $a, a' \in G$, $(a, a') \in H \Rightarrow (\pi(a), \pi(a')) \in H'$ and $(a, a') \in V \Rightarrow (\pi(a), \pi(a')) \in V'$. For instance, $\mathfrak{G}_{k \cdot m}$ is homomorphically embeddable into $\mathfrak{G}_m$ through reduction mod m . For a domino system $\mathcal{D}$, a *periodic tiling* is a morphism $\tau : \mathfrak{G}_m \rightarrow \mathcal{D}$ for some m and we say that $\mathcal{D}$ *admits a periodic tiling* if there exists a periodic tiling of $\mathcal{D}$.

The problem TILES (or *periodic tiling problem*), which is well known to be undecidable [12], is defined as follows: Given a domino system $\mathcal{D}$, does $\mathcal{D}$ admit a periodic tiling?

To use TILES in our reductions, we first use some specific bi-binary structures, which we call *grid-like* and which are easier to manipulate in our context to encode domino systems. A bi-binary structure $\mathfrak{G} = (A, H, V)$ is said to be *grid-like* if some $\mathfrak{G}_m$ is homomorphically embeddable into $\mathfrak{G}$. The logic FO *over bi-binary structures* refers to the first-order logic on two binary relations H, V , and we write Hxy to say that x and y are in relation for H . Consider the two following FO formulas over bi-binary structures: $\varphi_{\text{complete}} = \forall x. \forall y. \forall x'. \forall y'. ((Hxy \wedge \forall x x' \wedge \forall y y') \rightarrow Hx'y')$ and $\varphi_{\text{progress}} = \forall x. (\exists y. Hxy \wedge \exists y. Vxy)$. The following lemma, first stated and proved in [31], shows that these formulas suffice to characterize grid-like structures:

Lemma 10 ([31]). *Let $\mathfrak{G} = (A, H, V)$ be a bi-binary structure. If $\mathfrak{G}$ satisfies $\varphi_{\text{complete}}$ and $\varphi_{\text{progress}}$, then $\mathfrak{G}$ is grid-like.*

Given $\mathfrak{A} = (A, (P_\sigma), f_1, f_2,) \in \text{Data}[2, \Sigma]$ and $\varphi(x, y) \in \text{dFO}[2, \Sigma, \Gamma]$, we define the binary relation $\llbracket \varphi \rrbracket_{\mathfrak{A}} = \{(a, b) \in A \times A \mid \mathfrak{A} \models_{I[x/a][y/b]} \varphi(x, y) \text{ for some interpretation function } I\}$. Thus, given two $\text{dFO}[2, \Sigma, \Gamma]$ formulas $\varphi_1(x, y), \varphi_2(x, y)$ with two free variables, $(A, \llbracket \varphi_1 \rrbracket_{\mathfrak{A}}, \llbracket \varphi_2 \rrbracket_{\mathfrak{A}})$ is a bi-binary structure.

As we want to reason on data structures, we build a data structure $\mathfrak{A}_{2m}$ that corresponds to the grid $\mathfrak{G}_{2m} = (G_{2m}, H_{2m}, V_{2m})$. This structure is depicted locally in Figure 6. To define $\mathfrak{A}_{2m}$, we use four unary predicates given by $\Sigma_{\text{grid}} = \{X_0, X_1, Y_0, Y_1\}$. They give us access to the coordinate modulo 2. We then define $\mathfrak{A}_{2m} = (G_{2m}, f_1, f_2, (P_\sigma)) \in \text{Data}[2, \Sigma_{\text{grid}}]$ as follows: For $k \in \{0, 1\}$, we have $P_{X_k} = \{(i, j) \in G_{2m} \mid i \equiv k \bmod 2\}$ and $P_{Y_k} = \{(i, j) \in G_{2m} \mid j \equiv k \bmod 2\}$. For all $i, j \in \{0, \dots, 2m-1\}$, we set $f_1(i, j) = ((i/2) \bmod m) + m * ((j/2) \bmod m)$ (where $/$ stands for the Euclidian division). Finally, for all $i, j \in \{1, \dots, 2m\}$, set $f_2(i \bmod (2m), j \bmod (2m)) = f_1(i-1, j-1)$.

We provide below the definition of quantifier free formulas $\varphi_H(x, y)$ and $\varphi_V(x, y)$ from the logic $\text{dFO}[2, \Sigma_{\text{grid}}, \{1 \sim_1, 2 \sim_2\}]$ with two free variables. These formulas allow us to make

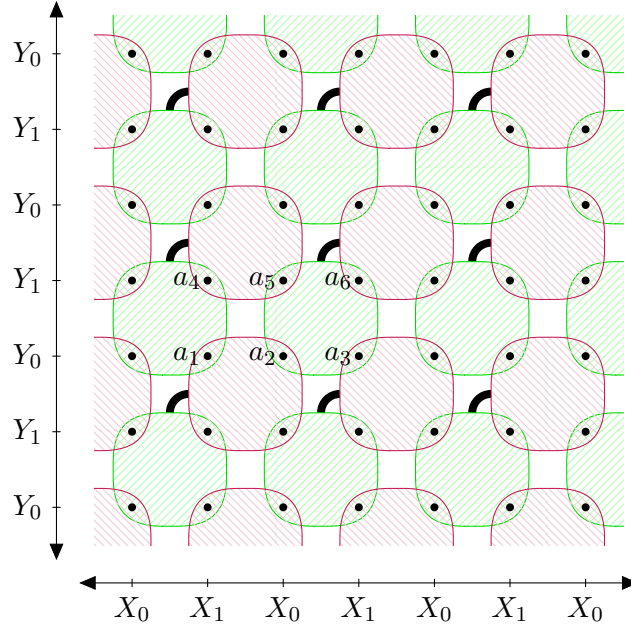


FIGURE 6. The local pattern of $\mathfrak{A}_{2m}$. Dots denote elements. Two dots are in the same $1 \sim_1$ -equivalence class (resp. $2 \sim_2$) iff they are in the same green (resp. purple) area. The thick black lines represent the relation $1 \sim_2$ in the following way: if a $1 \sim_1$ -equivalence class C_1 and a $2 \sim_2$ -equivalence class C_2 are connected with a thick black line, then for any $a \in C_1$ and $b \in C_2$, we have $a \sim_2 b$.

the link between the data structure $\mathfrak{A}_{2m}$ and the grid $\mathfrak{G}_{2m}$, and we will use them later on to ensure that a data structure has a shape 'similar' to $\mathfrak{A}_{2m}$.

$$\begin{aligned}
 \varphi_H^{00} &= X_0(x) \wedge X_1(y) \wedge Y_0(x) \wedge Y_0(y) \wedge x \sim_1 y \\
 \varphi_H^{10} &= X_1(x) \wedge X_0(y) \wedge Y_0(x) \wedge Y_0(y) \wedge x \sim_2 y \\
 \varphi_H^{01} &= X_0(x) \wedge X_1(y) \wedge Y_1(x) \wedge Y_1(y) \wedge x \sim_1 y \\
 \varphi_H^{11} &= X_1(x) \wedge X_0(y) \wedge Y_1(x) \wedge Y_1(y) \wedge x \sim_2 y \\
 \varphi_H &= \varphi_H^{00} \vee \varphi_H^{10} \vee \varphi_H^{01} \vee \varphi_H^{11} \\
 \varphi_V^{00} &= X_0(x) \wedge X_0(y) \wedge Y_0(x) \wedge Y_1(y) \wedge x \sim_1 y \\
 \varphi_V^{10} &= X_1(x) \wedge X_1(y) \wedge Y_0(x) \wedge Y_1(y) \wedge x \sim_1 y \\
 \varphi_V^{01} &= X_0(x) \wedge X_0(y) \wedge Y_1(x) \wedge Y_0(y) \wedge x \sim_2 y \\
 \varphi_V^{11} &= X_1(x) \wedge X_1(y) \wedge Y_1(x) \wedge Y_0(y) \wedge x \sim_2 y \\
 \varphi_V &= \varphi_V^{00} \vee \varphi_V^{10} \vee \varphi_V^{01} \vee \varphi_V^{11}
 \end{aligned}$$

Example 12. We draw six elements $a_1, a_2, a_3, a_4, a_5, a_6$ on Figure 6. According to the definition of $\mathfrak{A}_{2m}$ the data values associated to these elements are::

- $f_1(a_1) = m$ and $f_2(a_1) = 0$;

- $f_1(a_2) = m + 1$ and $f_2(a_2) = 0$;
- $f_1(a_3) = m + 1$ and $f_2(a_3) = 1$;
- $f_1(a_4) = m$ and $f_2(a_4) = m$;
- $f_1(a_5) = m + 1$ and $f_2(a_5) = m$;
- $f_1(a_6) = m + 1$ and $f_2(a_6) = m + 1$.

We have $f_1(a_1) = f_1(a_4)$ as a_1 and a_4 are in the same green area and $f_2(a_4) = f_2(a_5)$ as a_4 and a_5 are in the same purple area. Furthermore we have as well $f_1(a_4) = f_2(a_4)$ and a_4 lies in a green area connected to a purple area by a thick black line. We have then the following assertions: $\mathfrak{A}_{2m} \models \varphi_H^{10}(a_1, a_2)$, $\mathfrak{A}_{2m} \models \varphi_H^{00}(a_2, a_3)$, $\mathfrak{A}_{2m} \models \varphi_H^{11}(a_4, a_5)$, $\mathfrak{A}_{2m} \models \varphi_H^{01}(a_5, a_6)$, $\mathfrak{A}_{2m} \models \varphi_V^{10}(a_1, a_4)$, $\mathfrak{A}_{2m} \models \varphi_V^{00}(a_2, a_5)$ and $\mathfrak{A}_{2m} \models \varphi_V^{10}(a_3, a_6)$.

Using the definitions of G_{2m} and of $\mathfrak{A}_{2m}$ we can show the following lemma.

Lemma 11. *If $\mathfrak{G}$ is the bi-binary structure $(G_{2m}, \llbracket \varphi_H \rrbracket_{\mathfrak{A}_{2m}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}_{2m}})$, then $\mathfrak{G}_{2m} = \mathfrak{G}$.*

Proof. We have hence to prove that $H_{2m} = \{(a, b) \mid \mathfrak{A}_{2m} \models_{I[x/a][y/b]} \varphi_H(x, y) \text{ for some interpretation function } I\}$ and $V_{2m} = \{(a, b) \mid \mathfrak{A}_{2m} \models_{I[x/a][y/b]} \varphi_H(x, y) \text{ for some interpretation function } I\}$. We first show that $H_{2m} \subseteq \llbracket \varphi_H \rrbracket_{\mathfrak{A}_{2m}}$. Let $((i, j), (i', j')) \in H_{2m}$. Hence we have $j = j'$ and $i' - i \equiv 1 \pmod{2m}$. We have then different cases according to the parity of j, i and i' . Assume i, j are even. Then $(i, j), (i', j') \in P_{Y_0}$ and $(i, j) \in P_{X_0}$ and $(i', j) \in P_{X_1}$ and by definition of f_1 , we have $f_1(i, j) = f_1(i', j)$, hence $((i, j), (i', j)) \in \llbracket \varphi_H^{00} \rrbracket_{\mathfrak{A}_{2m}}$ and $((i, j), (i', j)) \in \llbracket \varphi_H \rrbracket_{\mathfrak{A}_{2m}}$. The other cases can be treated similarly.

We now prove that $H_{2m} \supseteq \llbracket \varphi_H \rrbracket_{\mathfrak{A}_{2m}}$. Let (a, b) be such that $\mathfrak{A}_{2m} \models_{I[x/a][y/b]} \varphi_H(x, y)$ for some interpretation function I . For φ_H to hold on (a, b) , one of the φ_H^{ij} must hold. We treat the case $\mathfrak{A}_{2m} \models_{I[x/a][y/b]} \varphi_H^{11}(a, b)$. Write (a_1, a_2) and (b_1, b_2) the coordinates of a and b respectively. As $a \in P_{X_0} \cap P_{Y_0}$ and $b \in P_{X_1} \cap P_{Y_0}$, we have that a_1, a_2, b_2 are even and b_1 is odd. As $a_1 \sim_1 b_1$, we have $((a_1/2) \bmod m) + m * ((a_2/2) \bmod m) = ((b_1/2) \bmod m) + m * ((b_2/2) \bmod m)$. This allows us to conclude that $a_2 = b_2$ and that $a_1 - b_1 \equiv 1 \pmod{m}$. So we have $(a, b) \in H_{2m}$. The other cases can be treated in a similar way.

The proof that $V_{2m} = \llbracket \varphi_V \rrbracket_{\mathfrak{A}_{2m}}$ follows the exact same lines. $\square$

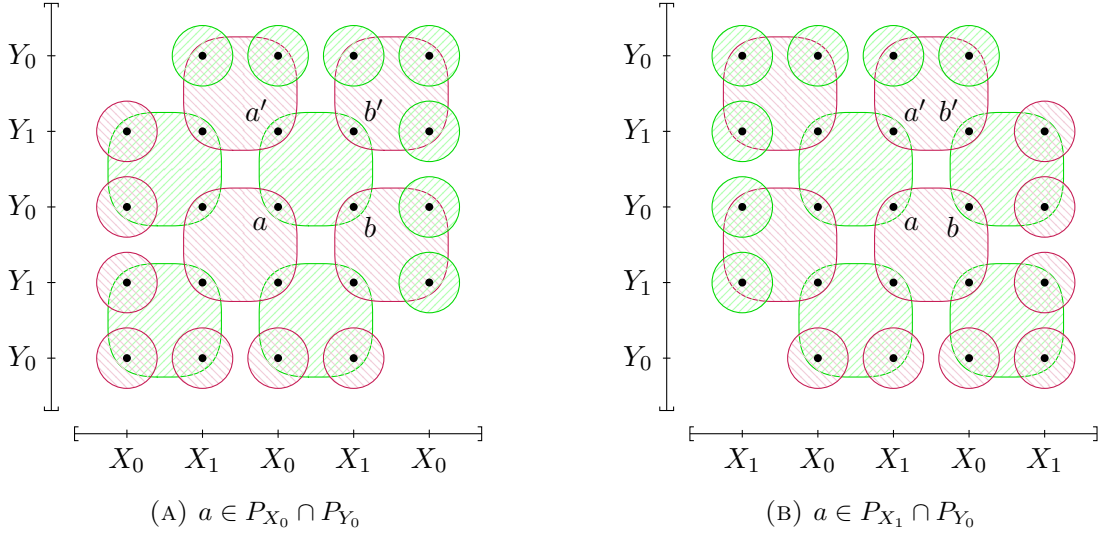
3.2.2. The Radius 3 case. We first use the previously introduced notions to show that $\text{DATASAT}(3\text{-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ is undecidable. Hence, we assume that $\Gamma = \{1 \sim_1, 2 \sim_2\}$. The first step in our reduction from TILES consists in defining the formula $\varphi_{\text{grid}}^{3\text{-loc}} \in 3\text{-Loc-dFO}[2, \Sigma_{\text{grid}}, \{1 \sim_1, 2 \sim_2\}]$ to check that a data structure corresponds to a grid ($\oplus$ stands for exclusive or):

$$\begin{aligned} \varphi_{\text{complete}}^{3\text{-loc}} &= \forall x. \langle \langle \forall y. \forall x'. \forall y'. \varphi_H(x, y) \wedge \varphi_V(x, x') \wedge \varphi_V(y, y') \rightarrow \varphi_H(x', y') \rangle \rangle_x^3 \\ \varphi_{\text{progress}}^{3\text{-loc}} &= \forall x. \langle \langle \exists y. \varphi_H(x, y) \wedge \exists y. \varphi_V(x, y) \rangle \rangle_x^3 \\ \varphi_{\text{grid}}^{3\text{-loc}} &= \varphi_{\text{complete}}^{3\text{-loc}} \wedge \varphi_{\text{progress}}^{3\text{-loc}} \wedge \forall x. \langle \langle (X_0(x) \oplus X_1(x)) \wedge (Y_0(x) \oplus Y_1(x)) \rangle \rangle_x^3 \end{aligned}$$

Lemma 12. *We have $\mathfrak{A}_{2m} \models \varphi_{\text{grid}}^{3\text{-loc}}$. Moreover, for all $\mathfrak{A} = (A, (P_\sigma), f_1, f_2)$ in $\text{Data}[2, \Sigma_{\text{grid}}]$, if $\mathfrak{A} \models \varphi_{\text{grid}}^{3\text{-loc}}$, then $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}})$ is grid-like.*

Proof. We first show that $\mathfrak{A}_{2m} \models \varphi_{\text{grid}}^{3\text{-loc}}$. In the proof, we assume that $m \geq 3$. The cases $m = 1$ or 2 are treated in the same way. Let us prove the first conjunct, that is $\mathfrak{A}_{2m} \models \varphi_{\text{complete}}^{3\text{-loc}}$. Let $a \in G_{2m}$. We want to prove that

$$\mathfrak{A}_{2m}|_a^3 \models_{I[x/a]} \forall y. \forall x'. \forall y'. \varphi_H(x, y) \wedge \varphi_V(x, x') \wedge \varphi_V(y, y') \Rightarrow \varphi_H(x', y')$$

FIGURE 7. Some 3-local views of $\mathfrak{A}_{2m}$ for $\Gamma = \{(1, 1), (2, 2)\}$.

for some interpretation function I . We fix an interpretation function I . We proceed by a case analysis on the values of $i, j \in \{0, 1\}$ such that $a \in P_{X_i} \cap P_{Y_j}$. Assume that $(i, j) = (0, 0)$. Then $\mathfrak{A}_{2m}|_a^3$ is depicted in Figure 7(A). Let b, a', b' such that

$$\mathfrak{A}_{2m}|_a^3 \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x, y) \wedge \varphi_V(x, x') \wedge \varphi_V(y, y').$$

We want to show

$$\mathfrak{A}_{2m}|_a^3 \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x', y').$$

By assumption on a and by looking at the definition of φ_H ,

$$\mathfrak{A}_{2m}|_a^3 \models_{I[x/a][y/b][x'/a'][y'/b']} X_1(y) \wedge Y_0(y) \wedge x \sim_1 y.$$

So by elimination we have that b is the element pointed by Figure 7a. In a similar way, a' and b' are indeed the elements pointed by Figure 7(B). Hence, we deduce

$$\mathfrak{A}_{2m}|_a^3 \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x', y').$$

The case $(i, j) = (1, 0)$ is depicted in Figure 7b and is proven in the same way just as the cases when $(i, j) = (1, 0)$ or $(i, j) = (1, 1)$.

Showing that $\mathfrak{A}_{2m} \models \varphi_{progress}^{3-loc}$ is done in the same way as showing that $\mathfrak{A}_{2m} \models \varphi_{complete}^{3-loc}$.

Finally, it is obvious that $\mathfrak{A}_{2m}$ satisfies the last conjunct of φ_{grid}^{3-loc} .

We now show that for all $\mathfrak{A} = (A, (P_\sigma), f_1, f_2)$ in $\text{Data}[\Sigma_{grid}]$, if $\mathfrak{A} \models \varphi_{grid}^{3-loc}$ then $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}})$ is grid-like. By Lemma 10, we just have to prove that $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}})$ satisfies $\varphi_{complete}$ and $\varphi_{progress}$. Let us prove that

$$(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}}) \models \forall x. \forall y. \forall x'. \forall y'. ((Hxy \wedge \forall x x' \wedge \forall y y') \Rightarrow Hx'y').$$

By definition of $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}})$, this amounts to verifying that

$$\mathfrak{A} \models \forall x. \forall y. \forall x'. \forall y'. \varphi_H(x, y) \wedge \varphi_V(x, x') \wedge \varphi_V(y, y') \Rightarrow \varphi_H(x', y').$$

Let $a, b, a', b' \in A$ and let I be an interpretation function such that $\mathfrak{A} \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x, y) \wedge \varphi_V(x, x') \wedge \varphi_V(y, y')$. Let us show $\mathfrak{A} \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x', y')$. We do a case analysis on $i, j \in \{0, 1\}$ such that $a \in P_{X_i} \cap P_{Y_j}$. We only perform the proof for the

case $(i, j) = (1, 0)$, the other three case can be treated similarly. By looking at φ_H and φ_V , we have

$$\begin{aligned}\mathfrak{A} &\models_{I[x/a][y/b][x'/a'][y'/b']} X_0(y) \wedge Y_0(y) \wedge x \sim_2 y \\ \mathfrak{A} &\models_{I[x/a][y/b][x'/a'][y'/b']} X_0(y') \wedge Y_1(y') \wedge y \sim_1 y' \\ \mathfrak{A} &\models_{I[x/a][y/b][x'/a'][y'/b']} X_1(x') \wedge Y_1(x') \wedge x \sim_1 x'.\end{aligned}$$

So b, a, b' are elements of $\mathfrak{A}_a^3$ and

$$\mathfrak{A}_a^3 \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x, y) \wedge \varphi_V(x, x') \wedge \varphi_V(y, y').$$

Since by assumption $\mathfrak{A} \models \varphi_{complete}^{3-loc}$, we deduce that $\mathfrak{A}_a^3 \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x', y')$. This allows us to conclude that $\mathfrak{A} \models_{I[x/a][y/b][x'/a'][y'/b']} \varphi_H(x', y')$.

We can prove in a similar way that $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}}) \models \varphi_{progress}$. $\square$

Given a domino system $\mathcal{D} = (D, H_{\mathcal{D}}, V_{\mathcal{D}})$, we now provide a formula $\varphi_{\mathcal{D}}$ from the logic 3-Loc-dFO[2, $D, \{1 \sim_1, 2 \sim_2\}$] that guarantees that, if a data structure corresponding to a grid satisfies $\varphi_{\mathcal{D}}$, then it can be embedded into $\mathcal{D}$:

$$\begin{aligned}\varphi_{\mathcal{D}} &:= \forall x. \langle \langle \bigvee_{d \in D} (d(x) \wedge \bigwedge_{d' \neq d \in D} \neg(d(x) \wedge d'(x))) \rangle \rangle_x^3 \\ &\quad \wedge \forall x. \langle \langle \forall y. \varphi_H(x, y) \rightarrow \bigvee_{(d, d') \in H_{\mathcal{D}}} d(x) \wedge d'(y) \rangle \rangle_x^3 \\ &\quad \wedge \forall x. \langle \langle \forall y. \varphi_V(x, y) \rightarrow \bigvee_{(d, d') \in V_{\mathcal{D}}} d(x) \wedge d'(y) \rangle \rangle_x^3\end{aligned}$$

Proposition 3. *Given $\mathcal{D} = (D, H_{\mathcal{D}}, V_{\mathcal{D}})$ a domino system, $\mathcal{D}$ admits a periodic tiling iff the 3-Loc-dFO[2, $\Sigma_{grid} \uplus D, \{1 \sim_1, 2 \sim_2\}$] formula $\varphi_{grid}^{3-loc} \wedge \varphi_{\mathcal{D}}$ is satisfiable.*

Proof. First assume that $\mathcal{D}$ admits a periodic tiling and let $\tau : \mathfrak{G}_m \rightarrow D$ be one. As with Lemma 12 we already have that $\mathfrak{A}_{2m} \models \varphi_{grid}^{3-loc}$. From $\mathfrak{A}_{2m}$ we build another data structure $\mathfrak{A}'_{2m} \in \text{Data}[\Sigma_{grid} \uplus D]$ by adding the predicates $(P_d)_{d \in D}$ as follow: for any $i, j \in \{0, 2m-1\}$ and $d \in D$ we set $P_d((i, j))$ to hold iff $\tau((i \bmod m, j \bmod m)) = d$. We can then show that $\mathfrak{A}_{2m} \models \varphi_{\mathcal{D}}$.

Assume now that there exists $\mathfrak{A} = (A, (P_{\sigma}), f_1, f_2)$ in $\text{Data}[\Sigma_{grid} \uplus D]$ such that $\mathfrak{A} \models \varphi_{grid}^{3-loc} \wedge \varphi_{\mathcal{D}}$. By Lemma 12, there exists $m > 0$ and a morphism $\pi : \mathfrak{G}_m \rightarrow (\mathfrak{A}, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}})$. It remains hence to show that there is a morphism $\tau : (\mathfrak{A}, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}}) \rightarrow \mathcal{D}$. For any $a \in A$, we set $\tau(a)$ to be a domino such that $P_{\tau(a)}(a)$ holds. Thanks to the first line of $\varphi_{\mathcal{D}}$, τ is well defined. Then thanks to the second and third line of $\varphi_{\mathcal{D}}$, we have that τ is a morphism. We deduce that $\tau \circ \pi$ is a periodic tiling of $\mathcal{D}$. $\square$

As a corollary of the proposition, we obtain the main result of this section.

Theorem 6. *DATASAT(3-Loc-dFO, 2, $\{1 \sim_1, 2 \sim_2\}$) is undecidable.*

3.2.3. The Radius 2 case. We can also reduce TILES to DATASAT(2-Loc-dFO, 2, $\{1 \sim_1, 2 \sim_2, 1 \sim_2\}$). In that case, it is a bit more subtle to build a formula similar to the formula $\varphi_{complete}$ as we have only neighborhood of radius 2. We use the diagonal binary relation $1 \sim_2$ to overcome this.

A *tri-binary structure* is a triple (A, H, V, W) where A is a set and H, V, W are three subsets of $A \times A$. Intuitively H, V will capture the horizontal and vertical adjacency relation whereas W will capture the diagonal adjacency. By an abuse of notation, $\mathfrak{G}_m$ will also refer

to the tri-binary structure (G_m, H_m, V_m, W_m) , where G_m, H_m and V_m are the same as before and:

$$W_m = \{((i, j), (i + 1, j + 1)) \mid i, j \in \mathbb{Z} \bmod m\}.$$

The logic FO *over tri-binary structures* is the same as FO over bi-binary structures with the addition of the binary symbol W. Let $\varphi'_{complete}$ be the following FO formula over tri-binary structure:

$$\varphi'_{complete} = \forall x. \forall y. \forall y'. (Hxy \wedge Vyy' \Rightarrow Wxy') \wedge \forall x. \forall x'. \forall y'. (Wxy' \wedge Vxx' \Rightarrow Hx'y').$$

Lemma 13. *Let $\mathfrak{G} = (A, H, V, W)$ be a tri-binary structure. If $\mathfrak{G}$ satisfies $\varphi'_{complete}$ and $\varphi_{progress}$, then (A, H, V) is grid-like.*

Proof. We simply remark that $\varphi'_{complete}$ implies $\varphi_{complete}$ and then we apply Lemma 10. $\square$

As in the previous subsection, we will consider data structures in $\text{Data}[2, \Sigma_{grid}]$ to encode domino systems and we will use $2\text{-Loc-dFO}[2, \Sigma_{grid}, \{1 \sim_1, 2 \sim_2, 1 \sim_2\}]$ formulas in order to ensure that the data structures are grid-like and that an embedding of a domino system in it is feasible. In the previous subsection, to ensure that a data structure is a grid, we used the fact that we could look in our logical formulas to neighborhood of radius 3 (cf formula $\varphi_{grid}^{3\text{-loc}}$), but since here we want to look at neighborhoods of radius 2, we use the diagonal relation and rely on the result of the previous lemma. Consequently, we will need again the two quantifier free formulas $\varphi_H(x, y)$ and $\varphi_V(x, y)$ of $\text{dFO}[2, \Sigma_{grid}, 1 \sim_1, 2 \sim_2]$ introduced previously and we define a new quantifier free formula $\varphi_W(x, y)$ in $\text{dFO}[2, \Sigma_{grid}, \{1 \sim_2\}]$:

$$\begin{aligned} \varphi_W^{00} &= X_0(x) \wedge X_1(y) \wedge Y_0(x) \wedge Y_1(y) \wedge x \sim_2 y \\ \varphi_W^{10} &= X_1(x) \wedge X_0(y) \wedge Y_0(x) \wedge Y_1(y) \wedge x \sim_2 y \\ \varphi_W^{01} &= X_0(x) \wedge X_1(y) \wedge Y_1(x) \wedge Y_0(y) \wedge x \sim_2 y \\ \varphi_W^{11} &= X_1(x) \wedge X_0(y) \wedge Y_1(x) \wedge Y_0(y) \wedge x \sim_2 y \\ \varphi_W &= \varphi_W^{00} \vee \varphi_W^{10} \vee \varphi_W^{01} \vee \varphi_W^{11} \end{aligned}$$

We will now define a formula $\varphi_{grid}^{2\text{-loc}}$ in $2\text{-Loc-dFO}[2, \Sigma_{grid}, \{1 \sim_1, 2 \sim_2, 1 \sim_2\}]$ which ensures that a data structure corresponds to a grid. This formula is given by ($\oplus$ stands for exclusive or):

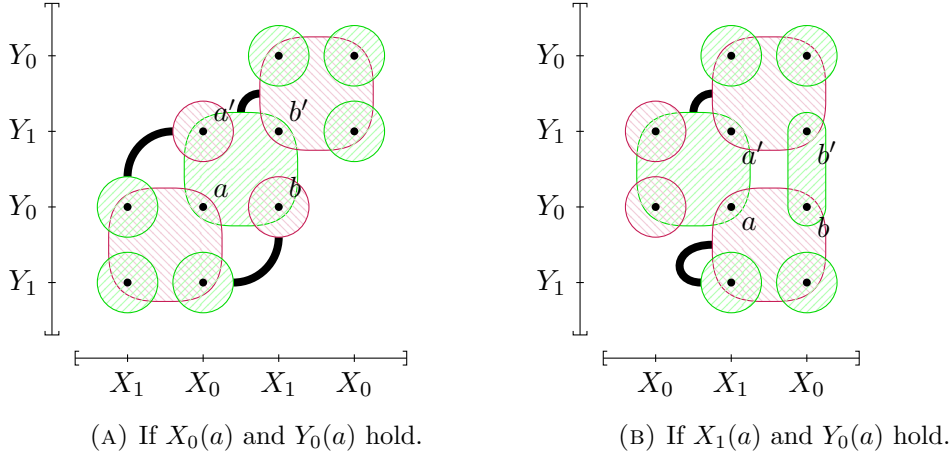
$$\begin{aligned} \varphi_{complete}^{2\text{-loc}} &= \forall x. \langle \langle \forall y y'. \varphi_H(x, y) \wedge \varphi_V(y, y') \Rightarrow \varphi_W(x, y') \rangle \rangle_x^2 \\ &\quad \wedge \forall x. \langle \langle \forall y y'. \varphi_V(x, y') \wedge \varphi_W(x, y') \Rightarrow \varphi_H(x', y') \rangle \rangle_x^2 \\ \varphi_{progress}^{2\text{-loc}} &= \forall x. \langle \langle \exists y. \varphi_H(x, y) \wedge \exists y. \varphi_V(x, y) \rangle \rangle_x^2 \\ \varphi_{grid}^{2\text{-loc}} &= \varphi_{complete}^{2\text{-loc}} \wedge \varphi_{progress}^{2\text{-loc}} \wedge \forall x. \langle \langle (X_0(x) \oplus X_1(x)) \wedge (Y_0(x) \oplus Y_1(x)) \rangle \rangle_x^2 \end{aligned}$$

We can then establish the following result.

Lemma 14. *The following statements hold:*

- (1) $\mathfrak{A}_{2m} \models \varphi_{grid}^{2\text{-loc}}$, and
- (2) for all $\mathfrak{A} = (A, f_1, f_2, (P_\sigma)) \in \text{Data}[\Sigma_{grid}]$, if $\mathfrak{A} \models \varphi_{grid}^{2\text{-loc}}$, then $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}})$ is grid-like.

Proof. The proof is similar to the of Lemma 12. For the first point, Figure 8 provides some representation of $\mathfrak{A}_{2m}|_a^2$ for some elements $a \in G_{2m}$. For instance, if we look at the case presented on Figure 8(A), we have that $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_H(x, y)$

FIGURE 8. Some 2-local views of $\mathfrak{A}_{2m}$ for $\Gamma = \{1 \sim_1, 2 \sim_2, 1 \sim_2\}$.

and $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_V(y, y')$ and $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_V(x, x'')$ and $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_W^{00}(x, y')$. Similarly, if we look at the case presented on Figure 8(B), we have that $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_H(x, y)$ and $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_V(y, y')$ and $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_V(x, x'')$ and $\mathfrak{A}_{2m}|_a^2 \models_{I[x/a][x'/a'][y/b][y'/b']} \varphi_W^{10}(x, y')$. For the second point of lemma, following the same reasoning as in Lemma 12.2, we first show that the tri-binary structure $(A, \llbracket \varphi_H \rrbracket_{\mathfrak{A}}, \llbracket \varphi_V \rrbracket_{\mathfrak{A}}, \llbracket \varphi_W \rrbracket_{\mathfrak{A}})$ satisfies $\varphi'_{complete}$ and $\varphi_{progress}$ and we use Lemma 13 to conclude. $\square$

As previously, we provide a formula φ'_D of $2\text{-Loc-dFO}[2, D, \{1 \sim_1, 2 \sim_2, 1 \sim_2\}]$ for any domino system $\mathcal{D} = (D, H_D, V_D)$. This formula is morally the same as the formula φ_D , we only restrict the neighborhood, but in fact this does not change anything:

$$\begin{aligned} \varphi'_D := & \forall x. \langle \langle \bigvee_{d \in D} d(x) \wedge \bigwedge_{d' \neq d \in D} \neg(d(x) \wedge d'(x)) \rangle \rangle_x^2 \\ & \wedge \forall x. \langle \langle \forall y. \varphi_H(x, y) \Rightarrow \bigvee_{(d, d') \in H_D} d(x) \wedge d'(y) \rangle \rangle_x^2 \\ & \wedge \forall x. \langle \langle \forall y. \varphi_V(x, y) \Rightarrow \bigvee_{(d, d') \in V_D} d(x) \wedge d'(y) \rangle \rangle_x^2 \end{aligned}$$

We have the following proposition whose proof follows the same line as Proposition 3.

Proposition 4. *Given $\mathcal{D} = (D, H_D, V_D)$ a domino system, we have that $\mathcal{D}$ admits a periodic tiling iff the $2\text{-Loc-dFO}[2, \Sigma_{grid} \uplus D, \{1 \sim_1, 2 \sim_2, 1 \sim_2\}]$ formula $\varphi_{grid}^{2\text{-loc}} \wedge \varphi'_D$ is satisfiable.*

Finally, we obtain the desired undecidability result.

Theorem 7. *DATASAT($2\text{-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2, 1 \sim_2\}$) is undecidable.*

4. EXISTENTIAL FRAGMENT

We present in this section results on the existential fragment of $r\text{-Loc-dFO}[Da, \Sigma, \Gamma_{Da}]$ (with $r \geq 1$ and $Da \geq 1$ and Γ_{Da} to represent the set of binary relation symbols $\{i \sim_j \mid i, j \in \{1, \dots, Da\}\}$) and establish when it is decidable. This fragment $\exists\text{-}r\text{-Loc-dFO}[Da, \Sigma, \Gamma_{Da}]$ is given by the grammar

$$\varphi ::= \langle \langle \psi \rangle \rangle_x^r \mid x = y \mid \neg(x = y) \mid \exists x. \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$$

where ψ is a formula from $\text{dFO}[Da, \Sigma, \Gamma_{Da}]$ with (at most) one free variable x . The quantifier free fragment $\text{qf-}r\text{-Loc-dFO}[Da, \Sigma, \Gamma_{Da}]$ is defined by the grammar $\varphi ::= \langle\langle\psi\rangle\rangle_x^r \mid x = y \mid \neg(x = y) \mid \varphi \vee \varphi \mid \varphi \wedge \varphi$.

Remark 2. *Note that for both these fragments, we do not impose any restrictions on the use of quantifiers in the formula ψ located under the bracket of the form $\langle\langle\psi\rangle\rangle_x^r$.*

4.1. Two Data Values and Balls of Radius 2.

We prove that the satisfiability problem for the existential fragment of local first-order logic with two data values and balls of radius two is decidable. To obtain this result we provide a reduction to the satisfiability problem for first-order logic over 1-data structures (see Theorem 3). Our reduction is based on the following intuition. Consider a 2-data structure $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Sigma]$ and an element $a \in A$. If we take an element b in $B_2^{\mathfrak{A}}(a)$, the radius-2-ball around a , we know that either $f_1(b)$ or $f_2(b)$ is a common value with a . In fact, if b is at distance 1 of a , this holds by definition and if b is at distance 2 then b shares an element with c at distance 1 of a and this element has to be shared with a as well so b ends to be at distance 1 of a . The trick consists then in using extra-labels for elements sharing a value with a that can be forgotten and to keep only the value of b not present in a , this construction leading to a 1-data structure. In order to have a sound and complete reduction, we need as well to show that we can enforce the 1-data structures that satisfied our formula to have a 'good' shape (i.e. they morally can be obtained from a 2-data structures by applying the aforementioned construction) and for this, we provide a formula of $\text{dFO}[1, \Sigma', \{1 \sim_1\}]$ (where Σ' is obtained from Σ by adding extra predicates).

The first step for our reduction consists in providing a characterization for the elements located in the radius-1-ball and the radius-2-ball around another element.

Lemma 15. *Let $\mathfrak{A} = (A, (P_\sigma), f_1, f_2) \in \text{Data}[2, \Sigma]$ and $a, b \in A$ and $j \in \{1, 2\}$. We have:*

- (1) $(b, j) \in B_1^{\mathfrak{A}}(a)$ iff there is $i \in \{1, 2\}$ such that $a \sim_j^{\mathfrak{A}} b$.
- (2) $(b, j) \in B_2^{\mathfrak{A}}(a)$ iff there exists $i, k \in \{1, 2\}$ such that $a \sim_k^{\mathfrak{A}} b$.

Proof. We show both statements:

- (1) Since $(b, j) \in B_1^{\mathfrak{A}}(a)$, by definition we have either $b = a$ and in that case $a \sim_j^{\mathfrak{A}} b$ holds, or $b \neq a$ and necessarily there exists $i \in \{1, 2\}$ such that $a \sim_j^{\mathfrak{A}} b$.
- (2) First, if there exists $i, k \in \{1, 2\}$ such that $a \sim_k^{\mathfrak{A}} b$, then $(b, k) \in B_1^{\mathfrak{A}}(a)$ and $(b, j) \in B_2^{\mathfrak{A}}(a)$ by definition. Assume now that $(b, j) \in B_2^{\mathfrak{A}}(a)$. Hence there exists $i \in \{1, 2\}$ such that $d^{\mathfrak{A}}((a, i), (b, j)) \leq 2$. We perform a case analysis on the value of $d^{\mathfrak{A}}((a, i), (b, j))$.
 - **Case** $d^{\mathfrak{A}}((a, i), (b, j)) = 0$. In that case $a = b$ and $i = j$ and we have $a \sim_i^{\mathfrak{A}} b$.
 - **Case** $d^{\mathfrak{A}}((a, i), (b, j)) = 1$. In that case, $((a, i), (b, j))$ is an edge in the data graph $\mathcal{G}(\mathfrak{A})$ of $\mathfrak{A}$ which means that $a \sim_j^{\mathfrak{A}} b$ holds.
 - **Case** $d^{\mathfrak{A}}((a, i), (b, j)) = 2$. Note that we have by definition $a \neq b$. Furthermore, in that case, there is $(c, k) \in A \times \{1, 2\}$ such that $((a, i), (c, k))$ and $((c, k), (b, j))$ are edges in $\mathcal{G}(\mathfrak{A})$. If $c \neq a$ and $c \neq b$, this implies that $a \sim_k^{\mathfrak{A}} c$ and $c \sim_j^{\mathfrak{A}} b$, so $a \sim_j^{\mathfrak{A}} b$ and $d^{\mathfrak{A}}((a, i), (b, j)) = 1$ which is a contradiction. If $c = a$ and $c \neq b$, this implies that $a \sim_k^{\mathfrak{A}} b$. If $c \neq a$ and $c = b$, this implies that $a \sim_k^{\mathfrak{A}} b$. $\square$

We consider a formula $\varphi = \exists x_1 \dots \exists x_n. \varphi_{qf}(x_1, \dots, x_n)$ of $\exists$ -2-Loc-dFO[2, Σ , Γ_2] in prenex normal form, i.e., such that $\varphi_{qf}(x_1, \dots, x_n) \in \text{qf-2-Loc-dFO}[2, \Sigma, \Gamma_2]$. We know that there is a structure $\mathfrak{A} = (A, (P_\sigma), f_1, f_2)$ in $\text{Data}[2, \Sigma]$ such that $\mathfrak{A} \models \varphi$ if and only if there are $a_1, \dots, a_n \in A$ such that $\mathfrak{A} \models \varphi_{qf}(a_1, \dots, a_n)$.

Let $\mathfrak{A} = (A, (P_\sigma), f_1, f_2)$ be a structure in $\text{Data}[2, \Sigma]$ and a tuple $\vec{a} = (a_1, \dots, a_n)$ of elements in A^n . We shall present the construction of a 1-data structure $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ in $\text{Data}[1, \Sigma']$ (with $\Sigma \subseteq \Sigma'$) with the same set of nodes as $\mathfrak{A}$, but where each node carries a single data value. In order to retrieve the data relations that hold in $\mathfrak{A}$ while reasoning over $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$, we introduce extra-predicates in Σ' to establish whether a node shares a common value with one of the nodes among $a_1, \dots, a_n$ in $\mathfrak{A}$.

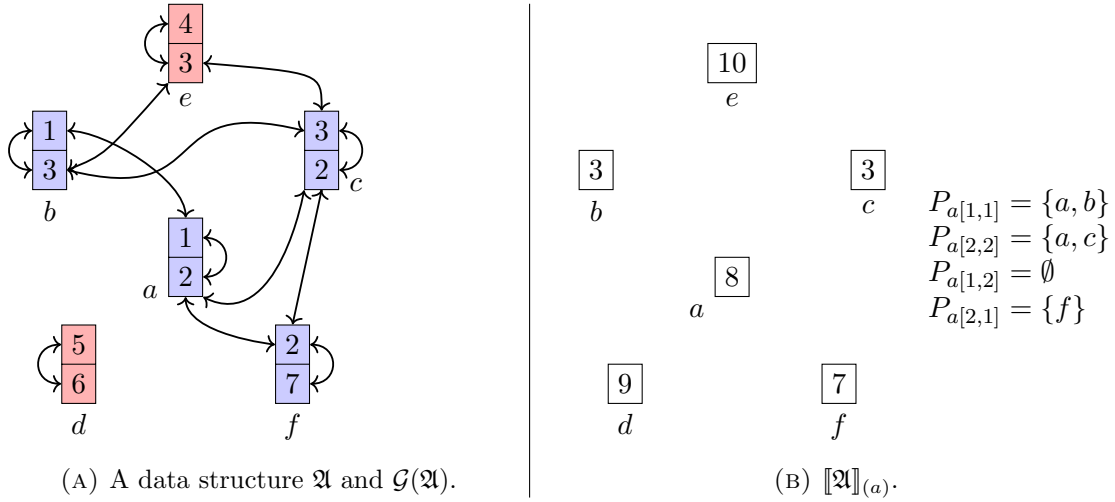


FIGURE 9

We now explain formally how we build $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$. Let $\Lambda_n = \{a_p[i, j] \mid p \in \{1, \dots, n\}, i, j \in \{1, 2\}\}$ be a set of new unary predicates and $\Sigma' = \Sigma \cup \Lambda_n$. For every element $b \in A$, the predicates in Λ_n are used to keep track of the relation between the data values of b and the one of $a_1, \dots, a_n$ in $\mathfrak{A}$. Formally, we define $P_{a_p[i, j]} = \{b \in A \mid \mathfrak{A} \models a_p \sim_j b\}$. We now define a data function $f : A \rightarrow \mathbb{N}$. We recall for this matter that $\text{Val}_{\mathfrak{A}}(\vec{a}) = \{f_1(a_1), f_2(a_1), \dots, f_1(a_n), f_2(a_n)\}$ and let $f_{\text{new}} : A \rightarrow \mathbb{N} \setminus \text{Val}_{\mathfrak{A}}(A)$ be an injection. For every $b \in A$, we set:

$$f(b) = \begin{cases} f_2(b) & \text{if } f_1(b) \in \text{Val}_{\mathfrak{A}}(\vec{a}) \text{ and } f_2(b) \notin \text{Val}_{\mathfrak{A}}(\vec{a}) \\ f_1(b) & \text{if } f_1(b) \notin \text{Val}_{\mathfrak{A}}(\vec{a}) \text{ and } f_2(b) \in \text{Val}_{\mathfrak{A}}(\vec{a}) \\ f_{\text{new}}(b) & \text{otherwise} \end{cases}$$

Hence depending if $f_1(b)$ or $f_2(b)$ is in $\text{Val}_{\mathfrak{A}}(\vec{a})$, it splits the elements of $\mathfrak{A}$ in four categories. If $f_1(b)$ and $f_2(b)$ are in $\text{Val}_{\mathfrak{A}}(\vec{a})$, the predicates in Λ_n allow us to retrieve all the data values of b . Given $j \in \{1, 2\}$, if $f_j(b)$ is in $\text{Val}_{\mathfrak{A}}(\vec{a})$ but $f_{3-j}(b)$ is not, the new predicates will give us the j -th data value of b and we have to keep track of the $(3-j)$ -th one, so we save it in $f(b)$. Lastly, if neither $f_1(b)$ nor $f_2(b)$ is in $\text{Val}_{\mathfrak{A}}(\vec{a})$, we will never be able to see the data values of b in φ_{qf} (thanks to Lemma 15), so they do not matter to us. Finally, we have $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} = (A, (P_\sigma)_{\sigma \in \Sigma'}, f)$. Figure 9b provides an example of $\text{Val}_{\mathfrak{A}}(\vec{a})$ for the data structures

depicted on Figure 9a and $\vec{a} = (a)$. The next lemma formalizes the connection existing between $\mathfrak{A}$ and $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ with $\vec{a} = (a_1, \dots, a_n)$.

Lemma 16. *Let $p \in \{1, \dots, n\}$ and assume $\mathfrak{A}|_{a_p}^2 = (A', (P'_\sigma), f_1^p, f_2^p)$ (with $A' = \{b \in A \mid (b, i) \in B_2^{\mathfrak{A}}(a_p)\}$). For all $b, c \in A'$ and $j, k \in \{1, 2\}$, the following statements hold:*

- (1) $(b, j), (c, k) \in B_2^{\mathfrak{A}}(a_p)$
- (2) If $(b, j) \in B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \in B_1^{\mathfrak{A}}(a_p)$ then $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ iff there is $i \in \{1, 2\}$ s.t. $b \in P_{a_p[i,j]}$ and $c \in P_{a_p[i,k]}$.
- (3) If $(b, j) \in B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \in B_1^{\mathfrak{A}}(a_p)$ then $\mathfrak{A}|_{a_p}^2 \not\models b \sim_k c$
- (4) If $(b, j), (c, k) \in B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$ then $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ iff either $b \sim_1^{\llbracket \mathfrak{A} \rrbracket_{\vec{a}}} c$ or there exists $p' \in \{1, \dots, n\}$ and $\ell \in \{1, 2\}$ such that $b \in P_{a_{p'}[\ell,j]}$ and $c \in P_{a_{p'}[\ell,k]}$.

Proof. (1) We show for instance that $(b, 1) \in B_2^{\mathfrak{A}}(a_p)$. Since b belongs to A' , it means that $(b, 2) \in B_2^{\mathfrak{A}}(a_p)$. Using Lemma 15.2, we have $(b, 1) \in B_1^{\mathfrak{A}}(a_p)$ or $(b, 2) \in B_1^{\mathfrak{A}}(a_p)$, hence $(b, 1) \in B_2^{\mathfrak{A}}(a_p)$.

(2) Assume that $(b, j) \in B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \in B_1^{\mathfrak{A}}(a_p)$. It implies that $f_j^p(b) = f_j(b)$ and $f_k^p(c) = f_k(c)$. Then assume that $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$. As $(b, j) \in B_1^{\mathfrak{A}}(a_p)$, thanks to Lemma 15.1 it means that there is a $i \in \{1, 2\}$ such that $a_p \sim_j^{\mathfrak{A}} b$. So we have $f_k(c) = f_k^p(c) = f_j^p(b) = f_j(b) = f_i(a_p)$, that is $a_p \sim_k^{\mathfrak{A}} c$. Hence by definition, $b \in P_{a_p[i,j]}$ and $c \in P_{a_p[i,k]}$. Conversely, let $i \in \{1, 2\}$ such that $b \in P_{a_p[i,j]}$ and $c \in P_{a_p[i,k]}$. This means that $a_p \sim_j^{\mathfrak{A}} b$ and $a_p \sim_k^{\mathfrak{A}} c$. So $f_j^p(b) = f_j(b) = f_i(a_p) = f_k(c) = f_k^p(c)$, that is $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$.

(3) Assume that $(b, j) \in B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \in B_1^{\mathfrak{A}}(a_p)$. It implies that $f_j^p(b) = f_j(b)$ and $f_k^p(c) = f_k(c)$. Thanks to Lemma 15.1, $(c, k) \in B_1^{\mathfrak{A}}(a_p)$ implies that $f_k(c) \in \{f_1(a_p), f_2(a_p)\}$ and $(b, j) \notin B_1^{\mathfrak{A}}(a_p)$ implies that $f_j(b) \notin \{f_1(a_p), f_2(a_p)\}$. So $\mathfrak{A}|_{a_p}^2 \not\models b \sim_k c$.

(4) Assume that $(b, j), (c, k) \in B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$. As previously, we have that $f_j(b) \notin \{f_1(a_p), f_2(a_p)\}$ and $f_k(c) \notin \{f_1(a_p), f_2(a_p)\}$, and thanks to Lemma 15.2, we have $f_{3-j}(b) \in \{f_1(a_p), f_2(a_p)\}$ and $f_{3-k}(c) \in \{f_1(a_p), f_2(a_p)\}$. There is then two cases:

- Suppose there does not exists $p' \in \{1, \dots, n\}$ such that $f_j(b) \in \{f_1(a_{p'}), f_2(a_{p'})\}$. This allows us to deduce that $f_j^p(b) = f_j(b) = f(b)$ and $f_k^p(c) = f_k(c)$. If $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$, then necessarily there does not exists $p' \in \{1, \dots, n\}$ such that $f_k(c) \in \{f_1(a_{p'}), f_2(a_{p'})\}$ so we have $f_k^p(c) = f_k(c) = f(c)$ and $f(b) = f(c)$, consequently $b \sim_1^{\llbracket \mathfrak{A} \rrbracket_{\vec{a}}} c$. Similarly assume that $b \sim_1^{\llbracket \mathfrak{A} \rrbracket_{\vec{a}}} c$, this means that $f(b) = f(c)$ and either $b = c$ and $k = j$ or $b \neq c$ and by injectivity of f , we have $f_j(b) = f(b) = f_k(c)$. This allows us to deduce that $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$.
- If there exists $p' \in \{1, \dots, n\}$ such that $f_j(b) = f_\ell(a_{p'})$ for some $\ell \in \{1, 2\}$. Then we have $b \in P_{a_{p'}[\ell,j]}$. Consequently, we have $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ iff $c \in P_{a_{p'}[\ell,k]}$. $\square$

We shall now see how we translate the formula $\varphi_{qf}(x_1, \dots, x_n)$ into a formula $\llbracket \varphi_{qf} \rrbracket(x_1, \dots, x_n)$ in $\text{dFO}[1, \Sigma', \{1 \sim 1\}]$ such that $\mathfrak{A}$ satisfies $\varphi_{qf}(a_1, \dots, a_n)$ if, and only if, $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ satisfies $\llbracket \varphi_{qf} \rrbracket(a_1, \dots, a_n)$. Thanks to the previous lemma we know that if $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ then (b, j) and (c, k) must belong to the same set among $B_1^{\mathfrak{A}}(a_p)$ and $B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$ and we can test in $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ whether (b, j) is a member of $B_1^{\mathfrak{A}}(a_p)$ or $B_2^{\mathfrak{A}}(a_p)$. Indeed, thanks to Lemmas 15.1 and 15.2, we have $(b, j) \in B_1^{\mathfrak{A}}(a_p)$ iff $b \in \bigcup_{i=1,2} P_{a_p[i,j]}$ and $(b, j) \in B_2^{\mathfrak{A}}(a_p)$ iff $b \in \bigcup_{i=1,2}^{j'=1,2} P_{a_p[i,j']}$.

This reasoning leads to the following formulas in $\text{dFO}[1, \Sigma', \{1 \sim_1\}]$ with $p \in \{1, \dots, n\}$ and $j \in \{1, 2\}$:

- $\varphi_{j, B_1(a_p)}(y) := a_p[1, j](y) \vee a_p[2, j](y)$ to test if the j -th field of an element belongs to $B_1^{\mathfrak{A}}(a_p)$
- $\varphi_{B_2(a_p)}(y) := \varphi_{1, B_1(a_p)}(y) \vee \varphi_{2, B_1(a_p)}(y)$ to test if a field of an element belongs to $B_2^{\mathfrak{A}}(a_p)$
- $\varphi_{j, B_2(a_p) \setminus B_1(a_p)}(y) := \varphi_{B_2(a_p)}(y) \wedge \neg \varphi_{j, B_1(a_p)}(y)$ to test that the j -th field of an element belongs to $B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$

We shall now present how we use these formulas to translate atomic formulas of the form $y \sim_k z$ under some $\langle\langle - \rangle\rangle_{x_p}^2$. For this matter, we rely on the two following formulas of $\text{dFO}[1, \Sigma']$:

- The first formula asks for (y, j) and (z, k) to be in $B_1^1(a_p)$ (where here we abuse notations, using variables for the elements they represent) and for these two data values to coincide with one data value of a_p , it corresponds to Lemma 16.2:

$$\varphi_{j, k, a_p}^{r=1}(y, z) := \varphi_{j, B_1(a_p)}(y) \wedge \varphi_{k, B_1(a_p)}(z) \wedge \bigvee_{i=1,2} a_p[i, j](y) \wedge a_p[i, k](z)$$

- The second formula asks for (y, j) and (z, k) to be in $B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$ and checks either whether the data values of y and z in $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ are equal or whether there exist p' and ℓ such that y belongs to $a_{p'}[\ell, j](y)$ and z belongs to $a_{p'}[\ell, k](z)$, it corresponds to Lemma 16.4:

$$\begin{aligned} \varphi_{j, k, a_p}^{r=2}(y, z) := & \varphi_{j, B_2(a_p) \setminus B_1(a_p)}(y) \wedge \varphi_{k, B_2(a_p) \setminus B_1(a_p)}(z) \wedge \\ & (y \sim z \vee (\bigvee_{p'=1}^n \bigvee_{\ell=1}^2 a_{p'}[\ell, j](y) \wedge a_{p'}[\ell, k](z))) \end{aligned}$$

Finally, here is the inductive definition of the translation $\llbracket - \rrbracket$ which uses sub transformations $\llbracket - \rrbracket_{x_p}$ in order to remember the centre of the ball and leads to the construction of $\llbracket \varphi_{qf} \rrbracket(x_1, \dots, x_n)$:

$$\begin{aligned} \llbracket \varphi \vee \varphi' \rrbracket &= \llbracket \varphi \rrbracket \vee \llbracket \varphi' \rrbracket \\ \llbracket x_p = x'_p \rrbracket &= x_p = x'_p \\ \llbracket \neg \varphi \rrbracket &= \neg \llbracket \varphi \rrbracket \\ \llbracket \langle\langle \psi \rangle\rangle_{x_p}^2 \rrbracket &= \llbracket \psi \rrbracket_{x_p} \\ \llbracket y \sim_k z \rrbracket_{x_p} &= \varphi_{j, k, a_p}^{r=1}(y, z) \vee \varphi_{j, k, a_p}^{r=2}(y, z) \\ \llbracket \sigma(x) \rrbracket_{x_p} &= \sigma(x) \\ \llbracket x = y \rrbracket_{x_p} &= x = y \\ \llbracket \varphi \vee \varphi' \rrbracket_{x_p} &= \llbracket \varphi \rrbracket_{x_p} \vee \llbracket \varphi' \rrbracket_{x_p} \\ \llbracket \neg \varphi \rrbracket_{x_p} &= \neg \llbracket \varphi \rrbracket_{x_p} \\ \llbracket \exists x. \varphi \rrbracket_{x_p} &= \exists x. \varphi_{B_2(a_p)}(x) \wedge \llbracket \varphi \rrbracket_{x_p} \end{aligned}$$

Lemma 17. *We have $\mathfrak{A} \models \varphi_{qf}(\vec{a})$ iff $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket \varphi_{qf} \rrbracket(\vec{a})$.*

Proof. Since $\llbracket \varphi_{qf} \rrbracket(x_1, \dots, x_n)$ belongs to the quantifier free fragment of $2\text{-Loc-dFO}[2, \Sigma, \Gamma_2]$, the only place where quantifiers can be used are in the local modalities $\langle\langle \psi \rangle\rangle_{x_p}^2$. As a matter of fact, because of the inductive definition of $\llbracket \varphi \rrbracket$ and that only the formulas $\exists x. \varphi$ and $y \sim_k z$ change, we only have to prove that $\mathfrak{A}|_{a_p}^2 \models \exists x. \varphi$ iff $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket \exists x. \varphi \rrbracket_{x_p}$ and that given $b, c \in A$, we have $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ iff $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket y \sim_k z \rrbracket_{x_p}(b, c)$.

To show that $\mathfrak{A}|_{a_p}^2 \models \exists x. \varphi$ iff $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket \exists x. \varphi \rrbracket_{x_p}$, we recall that $\mathfrak{A}|_{a_p}^2 = (A', (P'_\sigma), f'_1, \dots, f'_n)$ with $A' = \{b \in A \mid (b, i) \in B_r^{\mathfrak{A}}(a) \text{ for some } i \in \{1, 2\}\}$. Hence when we quantify existentially

in $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$, we should only consider the element belonging to A' . Thanks to Lemma 16.1, we know that if $b \in A'$ then $(b, 1)$ and $(b, 2)$ belong to $B_2^{\mathfrak{A}}(a_p)$. Consequently the formula $\varphi_{B_2(a_p)}(x)$ allows to focus on elements in A' .

We now prove that given $b, c \in A$, we have $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ iff $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket y \sim_k z \rrbracket_{x_p}(b, c)$ and we first suppose that $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$ and assume $\mathfrak{A}|_{a_p}^2 = (A', (P'_\sigma), f'_1, \dots, f'_n)$. Using Lemma 16, it implies that $b, c \in A'$ and (b, j) and (c, k) belong to same set between $B_1^{\mathfrak{A}}(a_p)$ and $B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$. We proceed by a case analysis.

- If $(b, j), (c, k) \in B_1^{\mathfrak{A}}(a_p)$ then by lemma 16.2 we have that $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \varphi_{j,k,a_p}^{r=1}(b, c)$ and thus $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket y \sim_k z \rrbracket_{x_p}(b, c)$.
- If $(b, j), (c, k) \in B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$ then by lemma 16.4 we have that $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \varphi_{j,k,a_p}^{r=2}(b, c)$ and thus $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket y \sim_k z \rrbracket_{x_p}(b, c)$.

We now suppose that $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \llbracket y \sim_k z \rrbracket_{x_p}(b, c)$. It means that $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ satisfies $\varphi_{j,k,a_p}^{r=1}(b, c)$ or $\varphi_{j,k,a_p}^{r=2}(b, c)$. If $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \varphi_{j,k,a_p}^{r=1}(b, c)$, it implies that (b, j) and (c, k) are in $B_1^{\mathfrak{A}}(a_p)$, and we can then apply lemma 16.2 to deduce that $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$. If $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \varphi_{j,k,a_p}^{r=2}(b, c)$, it implies that (b, j) and (c, k) are in $B_2^{\mathfrak{A}}(a_p) \setminus B_1^{\mathfrak{A}}(a_p)$, and we can then apply lemma 16.4 to deduce that $\mathfrak{A}|_{a_p}^2 \models b \sim_k c$. $\square$

To provide a reduction from $\text{DATASAT}(\exists\text{-2-Loc-dFO}, 2, \Gamma_2)$ to $\text{DATASAT}(\text{dFO}, 1, \Gamma_1)$, having the formula $\llbracket \varphi_{qf} \rrbracket(x_1, \dots, x_n)$ is not enough because to use the result of the previous lemma, we need to ensure that there exists a model $\mathfrak{B}$ and a tuple of elements $(a_1, \dots, a_n)$ such that $\mathfrak{B} \models \llbracket \varphi_{qf} \rrbracket(a_1, \dots, a_n)$ and as well that there exists $\mathfrak{A} \in \text{Data}[2, \Sigma]$ such that $\mathfrak{B} = \llbracket \mathfrak{A} \rrbracket_{\vec{a}}$. We explain now how we can ensure this last point.

Now, we want to characterize the structures of the form $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$. Given a structure $\mathfrak{B} = (A, (P_\sigma)_{\sigma \in \Sigma'}, f) \in \text{Data}[1, \Sigma']$ and $\vec{a} \in A$, we say that $(\mathfrak{B}, \vec{a})$ is *well formed* iff there exists a structure $\mathfrak{A} \in \text{Data}[2, \Sigma]$ such that $\mathfrak{B} = \llbracket \mathfrak{A} \rrbracket_{\vec{a}}$. Hence $(\mathfrak{B}, \vec{a})$ is *well formed* iff there exist two functions $f_1, f_2 : A \rightarrow \mathbb{N}$ such that $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} = \llbracket (A, (P_\sigma)_{\sigma \in \Sigma}, f_1, f_2) \rrbracket_{\vec{a}}$. We state three properties on $(\mathfrak{B}, \vec{a})$, and we will show that they characterize being well formed.

- (1) (Transitivity) For all $b, c \in A$, $p, q \in \{1, \dots, n\}$, $i, j, k, \ell \in \{1, 2\}$ if $b \in P_{a_p}[i, j]$, $c \in P_{a_p}[i, \ell]$ and $b \in P_{a_q}[k, j]$ then $c \in P_{a_q}[k, \ell]$.
- (2) (Reflexivity) For all p and i , we have $a_p \in P_{a_p}[i, i]$.
- (3) (Uniqueness) For all $b \in A$, if $b \in \bigcap_{j=1,2} \bigcup_{p=1, \dots, n}^{i=1,2} P_{a_p}[i, j]$ or $b \notin \bigcup_{j=1,2} \bigcup_{p=1, \dots, n}^{i=1,2} P_{a_p}[i, j]$ then for any $c \in B$ such that $f(c) = f(b)$ we have $c = b$.

Each property can be expressed by a first order logic formula, which we respectively name φ_{tran} , φ_{refl} and φ_{uniq} and we denote by φ_{wf} their conjunction:

$$\begin{aligned}
\varphi_{tran} &= \forall y \forall z. \bigwedge_{p,q=1}^n \bigwedge_{i,j,k,\ell=1}^2 \left(a_p[i, j](y) \wedge a_p[i, \ell](z) \wedge a_q[k, j](y) \Rightarrow a_q[k, \ell](z) \right) \\
\varphi_{refl}(x_1, \dots, x_n) &= \bigwedge_{p=1}^n \bigwedge_{i=1}^2 a_p[i, i](x_p) \\
\varphi_{uniq} &= \forall y. \left(\bigwedge_{j=1}^2 \bigvee_{p=1}^n \bigvee_{i=1}^2 a_p[i, j](y) \vee \bigwedge_{j=1}^2 \bigwedge_{p=1}^n \bigwedge_{i=1}^2 \neg a_p[i, j](y) \right) \Rightarrow \\
&\quad (\forall z. y \sim z \Rightarrow y = z) \\
\varphi_{wf}(x_1, \dots, x_n) &= \varphi_{tran} \wedge \varphi_{refl}(x_1, \dots, x_n) \wedge \varphi_{uniq}
\end{aligned}$$

The next lemma expresses that the formula φ_{wf} allows to characterise precisely the 1-data structures in $\text{Data}[1, \Sigma']$ which are well-formed.

Lemma 18. *Let $\mathfrak{B} \in \text{Data}[1, \Sigma']$ and $a_1, \dots, a_n$ elements of $\mathfrak{B}$, then $(\mathfrak{B}, \vec{a})$ is well formed iff $\mathfrak{B} \models \varphi_{wf}(\vec{a})$.*

Proof. First, if $(\mathfrak{B}, \vec{a})$ is well formed, then there exists $\mathfrak{A} \in \text{Data}[2, \Sigma]$ such that $\mathfrak{B} = \llbracket \mathfrak{A} \rrbracket_{\vec{a}}$ and by construction we have $\llbracket \mathfrak{A} \rrbracket_{\vec{a}} \models \varphi_{wf}(\vec{a})$. We now suppose that $\mathfrak{B} = (A, (P_\sigma)_{\sigma \in \Sigma'}, f)$ and $\mathfrak{B} \models \varphi_{wf}(\vec{a})$. In order to define the functions $f_1, f_2 : A \rightarrow \mathbb{N}$, we need to introduce some objects.

We first define a function $g : \{1, \dots, n\} \times \{1, 2\} \rightarrow \mathbb{N} \setminus \text{Im}(f)$ (where $\text{Im}(f)$ is the image of f in $\mathfrak{B}$) which verifies the following properties:

- for all $p \in \{1, \dots, n\}$ and $i \in \{1, 2\}$, we have $a_p \in P_{a_p[i, 3-i]}$ iff $g(p, 1) = g(p, 2)$;
- for all $p, q \in \{1, \dots, n\}$ and $i, j \in \{1, 2\}$, we have $a_p \in P_{a_p[i, j]}$ iff $g(p, i) = g(q, j)$.

We use this function to fix the two data values carried by the elements in $\{a_1, \dots, a_m\}$. We now explain why this function is well founded, it is due to the fact that $\mathfrak{B} \models \varphi_{tran} \wedge \varphi_{refl}(a_1, \dots, a_n)$. In fact, since $\mathfrak{B} \models \varphi_{refl}(a_1, \dots, a_n)$, we have for all $p \in \{1, \dots, n\}$ and $i \in \{1, 2\}$, $a_p \in P_{a_p[i, i]}$. Furthermore if $a_p \in P_{a_p[i, j]}$ then $a_p \in P_{a_p[j, i]}$ thanks to the formula φ_{tran} ; indeed since we have $a_p \in P_{a_p[i, j]}$ and $a_p \in P_{a_p[i, i]}$ and $a_p \in P_{a_p[j, j]}$, we obtain $a_p \in P_{a_p[j, i]}$. Next, we also have that if $a_q \in P_{a_p[i, j]}$ then $a_p \in P_{a_q[j, i]}$ again thanks to φ_{tran} ; indeed since we have $a_q \in P_{a_p[i, j]}$ and $a_p \in P_{a_p[i, i]}$ and $a_q \in P_{a_q[j, j]}$, we obtain $a_p \in P_{a_q[j, i]}$.

We also need a natural d_{out} belonging to $\mathbb{N} \setminus (\text{Im}(g) \cup \text{Im}(f))$. For $j \in \{1, 2\}$, we define f_j as follows for all $b \in A$:

$$f_j(b) = \begin{cases} g(p, i) & \text{if for some } p, i \text{ we have } b \in P_{a_p[i, j]} \\ f(b) & \text{if for all } p, i \text{ we have } b \notin P_{a_p[i, j]} \text{ and for some } p, i \text{ we have } b \in P_{a_p[i, 3-j]} \\ d_{out} & \text{if for all } p, i, j', \text{ we have } b \notin P_{a_p[i, j']} \end{cases}$$

Here again, we can show that since $\mathfrak{B} \models \varphi_{tran} \wedge \varphi_{refl}(a_1, \dots, a_n)$, the functions f_1 and f_2 are well founded. Indeed, assume that $b \in P_{a_p[i, j]} \cap P_{a_q[k, j]}$, then we have necessarily that $g(p, i) = g(q, k)$. For this we need to show that $a_p \in a_q[k, i]$ and we use again the formula φ_{tran} . This can be obtained because we have $b \in P_{a_p[i, j]}$ and $a_p \in P_{a_p[i, i]}$ and $b \in P_{a_q[k, j]}$.

We then define $\mathfrak{A}$ as the 2-data-structures $(A, (P_\sigma)_{\sigma \in \Sigma}, f_1, f_2)$. It remains to prove that $\mathfrak{B} = \llbracket \mathfrak{A} \rrbracket_{\vec{a}}$.

First, note that for all $b \in A$, $p \in \{1, \dots, n\}$ and $i, j \in \{1, 2\}$, we have $b \in P_{a_p[i, j]}$ iff $a_p \sim_j^{\mathfrak{A}} b$. Indeed, we have $b \in P_{a_p[i, j]}$, we have that $f_j(b) = g(p, i)$ and since $a_p \in P_{a_p[i, j]}$ we have as well that $f_i(a_p) = g(p, i)$, as a consequence $a_p \sim_j^{\mathfrak{A}} b$. In the other direction, if $a_p \sim_j^{\mathfrak{A}} b$, it means that $f_j(b) = f_i(a_p) = g(p, i)$ and thus $b \in P_{a_p[i, j]}$. Now to have $\mathfrak{B} = \llbracket \mathfrak{A} \rrbracket_{\vec{a}}$, one has only to be careful in the choice of function f_{new} while building $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}$. We recall that this function is injective and is used to give a value to the elements $b \in A$ such that neither $f_1(b) \in \text{Val}_{\mathfrak{A}}(\vec{a})$ and $f_2(b) \notin \text{Val}_{\mathfrak{A}}(\vec{a})$ nor $f_1(b) \notin \text{Val}_{\mathfrak{A}}(\vec{a})$ and $f_2(b) \in \text{Val}_{\mathfrak{A}}(\vec{a})$. For these elements, we make f_{new} matches with the function f and the fact that we define an injection is guaranteed by the formula φ_{uniq} . $\square$

Using the results of Lemma 17 and 18, we deduce that the formula $\varphi = \exists x_1 \dots \exists x_n. \varphi_{qf}(x_1, \dots, x_n)$ of $\exists$ -2-Loc-dFO[2, Σ, Γ_2] is satisfiable iff the formula $\psi = \exists x_1 \dots \exists x_n. \llbracket \varphi_{qf} \rrbracket(x_1, \dots, x_n) \wedge \varphi_{wf}(x_1, \dots, x_n)$ is satisfiable. Note that ψ can be built in polynomial time from φ and that it belongs to dFO[1, Σ', Γ_1]. Hence, thanks to Theorem 3, we obtain that DATASAT($\exists$ -2-Loc-dFO, 2) is in N2EXP.

We can as well obtain a matching lower bound thanks to a reduction from the problem DATASAT(dFO, 1, Γ_1). For this matter we rely on two crucial points. First in the formulas

of $\exists$ -2-Loc-dFO[2, Σ , Γ_2], there is no restriction on the use of quantifiers for the formulas located under the scope of the $\langle\langle \cdot \rangle\rangle_x^2$ modality and consequently we can write inside this modality a formula of dFO[1, Σ] without any modification. Second we can extend a model of dFO[1, Σ , Γ_1] into a 2-data structure such that all elements and their values are located in the same radius-2-ball by adding everywhere a second data value equal to 0. More formally, let φ be a formula in dFO[1, Σ , Γ_1] and consider the formula $\exists x. \langle\langle \varphi \rangle\rangle_x^2$ where we interpret φ over 2-data structures (this formula simply never mentions the values located in the second fields). We have then the following lemma.

Lemma 19. *There exists $\mathfrak{A} \in \text{Data}[1, \Sigma]$ such that $\mathfrak{A} \models \varphi$ if and only if there exists $\mathfrak{B} \in \text{Data}[2, \Sigma]$ such that $\mathfrak{B} \models \exists x. \langle\langle \varphi \rangle\rangle_x^2$.*

Proof. Assume that there exists $\mathfrak{A} = (A, (P_\sigma)_{\sigma \in \Sigma}, f_1)$ in $\text{Data}[1, \Sigma]$ such that $\mathfrak{A} \models \varphi$. Consider the 2-data structure $\mathfrak{B} = (A, (P_\sigma)_{\sigma \in \Sigma}, f_1, f_2)$ such that $f_2(a) = 0$ for all $a \in A$. Let $a \in A$. It is clear that we have $\mathfrak{B}|_a^2 = \mathfrak{B}$ and that $\mathfrak{B}|_a^2 \models \varphi$ (because $\mathfrak{A} \models \varphi$ and φ never mentions the second values of the elements since it is a formula in dFO[1, Σ]). Consequently $\mathfrak{B} \models \exists x. \langle\langle \varphi \rangle\rangle_x^2$.

Assume now that there exists $\mathfrak{B} = (A, (P_\sigma), f_1, f_2)$ in $\text{Data}[2, \Sigma]$ such that $\mathfrak{B} \models \exists x. \langle\langle \varphi \rangle\rangle_x^2$. Hence there exists $a \in A$ such that $\mathfrak{B}|_a^2 \models \varphi$, but then by forgetting the second value in $\mathfrak{B}|_a^2$ we obtain a model in $\text{Data}[1, \Sigma]$ which satisfies φ . $\square$

Since $\text{DATASAT}(\text{dFO}, 1, \Gamma_1)$ is N2EXP-hard (see Theorem 3), we obtain the desired lower bound.

Theorem 8. *The problem $\text{DATASAT}(\exists$ -2-Loc-dFO, 2, Γ_2) is N2EXP-complete.*

4.2. Balls of radius 1 and any number of data values.

Let $Da \geq 1$. We first show that $\text{DATASAT}(\exists$ -1-Loc-dFO, Da , Γ_{Da}) is in NEXP by providing a reduction towards $\text{DATASAT}(\text{dFO}, 0, \emptyset)$. This reduction uses the characterization of the radius-1-ball provided by Lemma 15 and is very similar to the reduction provided in the previous section. In fact, for an element b located in the radius-1-ball of another element a , we use extra unary predicates to explicitly characterise which are the values of b that are common with the values of a . We provide here the main step of this reduction whose proof follows the same line as the one of Theorem 8.

We consider a formula $\varphi = \exists x_1 \dots \exists x_n. \varphi_{qf}(x_1, \dots, x_n)$ of $\exists$ -1-Loc-dFO[Da , Σ , Γ_1] in prenex normal form, i.e., such that $\varphi_{qf}(x_1, \dots, x_n) \in \text{qf-1-Loc-dFO}[Da, \Sigma, \Gamma_1]$. We know that there is a structure $\mathfrak{A} = (A, (P_\sigma), f_1, f_2, \dots, f_{Da})$ in $\text{Data}[Da, \Sigma]$ such that $\mathfrak{A} \models \varphi$ if and only if there are $a_1, \dots, a_n \in A$ such that $\mathfrak{A} \models \varphi_{qf}(a_1, \dots, a_n)$. Let then $\mathfrak{A} = (A, (P_\sigma), f_1, f_2, \dots, f_{Da})$ in $\text{Data}[Da, \Sigma]$ and a tuple $\vec{a} = (a_1, \dots, a_n)$ of elements in A^n . Let $\Omega_n = \{a_p[i, j] \mid p \in \{1, \dots, n\}, i, j \in \{1, \dots, Da\}\}$ be a set of new unary predicates and $\Sigma' = \Sigma \cup \Omega_n$. For every element $b \in A$, the predicates in Ω_n are used to keep track of the relation between the data values of b and the ones of $a_1, \dots, a_n$ in $\mathfrak{A}$. Formally, we have $P_{a_p[i, j]} = \{b \in A \mid \mathfrak{A} \models a_p \sim_j b\}$. Finally, we build the 0-data-structure $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}' = (A, (P_\sigma)_{\sigma \in \Sigma'})$. Similarly to Lemma 16, we have the following connection between $\mathfrak{A}$ and $\llbracket \mathfrak{A} \rrbracket_{\vec{a}}'$.

Lemma 20. *Let $p \in \{1, \dots, n\}$ and assume $\mathfrak{A}|_{a_p}^1 = (A', (P'_\sigma), f_1, f_2, \dots, f_{Da})$ (with $A' = \{b \in A \mid (b, i) \in B_1^{\mathfrak{A}}(a_p)\}$) For all $b, c \in A'$ and $j, k \in \{1, \dots, Da\}$, the following statements hold:*

- (1) If $(b, j) \in B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \in B_1^{\mathfrak{A}}(a_p)$ then $\mathfrak{A}|_{a_p}^1 \models b \sim_k c$ iff there is $i \in \{1, \dots, Da\}$ s.t. $b \in P_{a_p[i,j]}$ and $c \in P_{a_p[i,k]}$.
- (2) If $(b, j) \notin B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \in B_1^{\mathfrak{A}}(a_p)$ then $\mathfrak{A}|_{a_p}^1 \not\models b \sim_k c$
- (3) If $(b, j) \notin B_1^{\mathfrak{A}}(a_p)$ and $(c, k) \notin B_1^{\mathfrak{A}}(a_p)$ then $\mathfrak{A}|_{a_p}^1 \models b \sim_k c$ iff $b = c$ and $j = k$.

We now translate $\varphi_{qf}(x_1, \dots, x_n)$ into a formula $\llbracket \varphi_{qf} \rrbracket'(x_1, \dots, x_n)$ in $\text{dFO}[0, \Sigma', \emptyset]$ such that $\mathfrak{A}$ satisfies $\varphi_{qf}(a_1, \dots, a_n)$ if, and only if, $\llbracket \mathfrak{A} \rrbracket'_{\vec{a}}$ satisfies $\llbracket \varphi_{qf} \rrbracket'(a_1, \dots, a_n)$. As in the previous section, we introduce the two following formulas in $\text{dFO}[0, \Sigma', \emptyset]$ with $p \in \{1, \dots, n\}$ and $j \in \{1, \dots, Da\}$ to:

- $\varphi_{j, B_1(a_p)}(y) := \bigvee_{i \in \{1, \dots, Da\}} a_p[i, j](y)$ to test if the j -th field of an element belongs to $B_1^{\mathfrak{A}}(a_p)$
- $\varphi_{B_1(a_p)}(y) := \bigvee_{j \in \{1, \dots, Da\}} \varphi_{j, B_1(a_p)}(y)$ to test if a field of an element belongs to $B_1^{\mathfrak{A}}(a_p)$

We now present how we translate atomic formulas of the form $y \sim_k z$ under some $\llbracket - \rrbracket'_{x_p}$. For this matter, we rely on two formulas of $\text{dFO}[0, \Sigma', \emptyset]$ which can be described as follows:

- The first formula asks for (y, j) and (z, k) to be in $B_1^{\mathfrak{A}}(a_p)$ (here we abuse notations, using variables for the elements they represent) and for these two data values to coincide with one data value of a_p , it corresponds to Lemma 20.1:

$$\psi_{j,k,a_p}^{r=1}(y, z) := \varphi_{j, B_1(a_p)}(y) \wedge \varphi_{k, B_1(a_p)}(z) \wedge \bigvee_{i=1}^{Da} a_p[i, j](y) \wedge a_p[i, k](z)$$

- The second formula asks for (y, j) and (z, k) to not belong to $B_1^{\mathfrak{A}}(a_p)$ and for $y = z$, it corresponds to Lemma 20.3:

$$\psi_{j,k,a_p}^{r>1}(y, z) := \begin{cases} \bigwedge_{i=1}^{Da} (\neg \varphi_{i, B_1(a_p)}(y) \wedge \neg \varphi_{i, B_1(a_p)}(z)) \wedge y = z & \text{if } j = k \\ \perp & \text{otherwise} \end{cases}$$

Finally, as before we provide an inductive definition of the translation $\llbracket - \rrbracket'$ which uses subtransformations $\llbracket - \rrbracket'_{x_p}$ in order to remember the centre of the ball and leads to the construction of $\llbracket \varphi_{qf} \rrbracket'(x_1, \dots, x_n)$. We only detail the cases:

$$\begin{aligned} \llbracket y \sim_k z \rrbracket'_{x_p} &= \psi_{j,k,a_p}^{r=1}(y, z) \vee \psi_{j,k,a_p}^{r>1}(y, z) \\ \llbracket \exists x. \varphi \rrbracket'_{x_p} &= \exists x. \varphi_{B_1(a_p)}(x) \wedge \llbracket \varphi \rrbracket'_{x_p} \end{aligned}$$

as the other cases are identical as for the translation $\llbracket - \rrbracket$ shown in the previous section. This leads to the following lemma (which is the pendant of Lemma 17).

Lemma 21. *We have $\mathfrak{A} \models \varphi_{qf}(\vec{a})$ iff $\llbracket \mathfrak{A} \rrbracket'_{\vec{a}} \models \llbracket \varphi_{qf} \rrbracket'(\vec{a})$.*

As we had to characterise the well-formed 1-data structure, a similar trick is necessary here. For this matter, we use the following formulas:

$$\begin{aligned} \psi_{tran} &= \forall y \forall z. \bigwedge_{p,q=1}^n \bigwedge_{i,j,k,\ell=1}^D \left(a_p[i, j](y) \wedge a_p[i, \ell](z) \wedge a_q[k, j](y) \Rightarrow a_q[k, \ell](z) \right) \\ \psi_{refl}(x_1, \dots, x_n) &= \bigwedge_{p=1}^n \bigwedge_{i=1}^D a_p[i, i](x_p) \\ \psi_{wf}(x_1, \dots, x_n) &= \psi_{tran} \wedge \psi_{refl}(x_1, \dots, x_n) \end{aligned}$$

Finally with the same reasoning as the one given in the previous section, we can show that the formula $\varphi = \exists x_1 \dots \exists x_n. \varphi_{qf}(x_1, \dots, x_n)$ of $\exists\text{-1-Loc-dFO}[D, \Sigma, \Gamma_{Da}]$ is satisfiable iff the formula $\exists x_1 \dots \exists x_n. \llbracket \varphi_{qf} \rrbracket'(x_1, \dots, x_n) \wedge \psi_{wf}(x_1, \dots, x_n)$ is satisfiable. Note that this

latter formula can be built in polynomial time from φ and that it belongs to $\text{dFO}[0, \Sigma', \emptyset]$. Hence, thanks to Theorem 2, we obtain that $\text{DATASAT}(\exists\text{-1-Loc-dFO}, Da, \Gamma_{Da})$ is in NEXP. The matching lower bound is as well obtained the same way by reducing $\text{DATASAT}(\text{dFO}, 0)$ to $\text{DATASAT}(\exists\text{-1-Loc-dFO}, Da, \Gamma_{Da})$ showing that a formula φ in $\text{dFO}[0, \Sigma, \emptyset]$ is satisfiable iff the formula $\exists x. \langle\langle\varphi\rangle\rangle_x^1$ in $\exists\text{-1-Loc-dFO}[Da, \Sigma, \Gamma_{Da}]$ is satisfiable.

Theorem 9. *For all $Da \geq 1$, the problem $\text{DATASAT}(\exists\text{-1-Loc-dFO}, Da, \Gamma_{Da})$ is NEXP-complete.*

4.3. Undecidability results.

We show here $\text{DATASAT}(\exists\text{-3-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ and $\text{DATASAT}(\exists\text{-2-Loc-dFO}, 3, \Gamma_3)$ are undecidable. To obtain this we provide reductions from $\text{DATASAT}(\text{dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ and we use the fact that any 2-data structure can be interpreted as a radius-3-ball of a 2-data structure or respectively as a radius-2-ball of a 3-data structure.

4.3.1. Radius 3 and two data values. In order to reduce $\text{DATASAT}(\text{dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ to the problem $\text{DATASAT}(\exists\text{-3-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2\})$, we show that we can transform any 2-data structure $\mathfrak{A}$ into another 2-data structure $\mathfrak{A}_{\text{ge}}$ such that $\mathfrak{A}_{\text{ge}}$ corresponds to the radius-3-ball of any element of $\mathfrak{A}_{\text{ge}}$ and this transformation has some kind of inverse. To obtain $\mathfrak{A}_{\text{ge}}$ from $\mathfrak{A}$, we add some elements and in order to not take them into account for the satisfaction of the formula, we label them with a fresh unary predicate ge . Furthermore, given a formula $\varphi \in \text{dFO}[2, \Sigma, \Gamma_2]$, we transform it into a formula $T(\varphi)$ in $\exists\text{-3-Loc-dFO}[2, \Sigma', \Gamma_2]$ such that $\mathfrak{A}$ satisfies φ iff $\mathfrak{A}_{\text{ge}}$ satisfies $T(\varphi)$. What follows is the formalization of this reasoning.

Let $\mathfrak{A} = (A, (P_\sigma)_\sigma, f_1, f_2)$ be a 2-data structure in $\text{Data}[2, \Sigma]$ and ge be a fresh unary predicate not in Σ . From $\mathfrak{A}$ we build the following 2-data structure $\mathfrak{A}_{\text{ge}} = (A', (P'_\sigma)_\sigma, f'_1, f'_2) \in \text{Data}[2, \Sigma \cup \{\text{ge}\}]$ such that:

- $A' = A \uplus \text{Val}_{\mathfrak{A}}(A) \times \text{Val}_{\mathfrak{A}}(A)$,
- for $i \in \{1, 2\}$ and $a \in A$, $f'_i(a) = f_i(a)$ and for $(d_1, d_2) \in \text{Val}_{\mathfrak{A}}(A) \times \text{Val}_{\mathfrak{A}}(A)$, $f_i((d_1, d_2)) = d_i$,
- for $\sigma \in \Sigma$, $P'_\sigma = P_\sigma$,
- $P_{\text{ge}} = \text{Val}_{\mathfrak{A}}(A) \times \text{Val}_{\mathfrak{A}}(A)$.

Hence to build $\mathfrak{A}_{\text{ge}}$ from $\mathfrak{A}$ we have added to the elements of $\mathfrak{A}$ all pairs of data presented in $\mathfrak{A}$ and in order to recognize these new elements in the structure we use the new unary predicate ge . We add these extra elements to ensure that all the elements of the structure are located in the radius-3-ball of any element of $\mathfrak{A}_{\text{ge}}$. We have then the following property.

Lemma 22. $|\mathfrak{A}_{\text{ge}}|_a^3 = \mathfrak{A}_{\text{ge}}$ for all $a \in A'$.

Proof. Let $b \in A'$ and $i, j \in \{1, 2\}$. We show that $d^{\mathfrak{A}_{\text{ge}}}((a, i), (b, j)) \leq 3$. i.e. that there is a path of length at most 3 from (a, i) to (b, j) in the data graph $\mathcal{G}(\mathfrak{A}_{\text{ge}})$. By construction of $\mathfrak{A}_{\text{ge}}$, there is an element $c \in A'$ such that $f_1(c) = f_i(a)$ and $f_2(c) = f_j(b)$. So we have the path $(a, i), (c, 1), (c, 2), (b, j)$ of length at most 3 from (a, i) to (b, j) in $\mathcal{G}(\mathfrak{A}_{\text{ge}})$. $\square$

Conversely, to $\mathfrak{A} = (A, (P_\sigma)_\sigma, f_1, f_2) \in \text{Data}[2, \Sigma \cup \{\text{ge}\}]$, we associate $\mathfrak{A}_{\setminus \text{ge}} = (A', (P'_\sigma)_\sigma, f'_1, f'_2) \in \text{Data}[2, \Sigma]$ where:

- $A' = A \setminus P_{\text{ge}}$,

- for $i \in \{1, 2\}$ and $a \in A'$, $f'_i(a) = f_i(a)$,
- for $\sigma \in \Sigma$, $P'_\sigma = P_\sigma \setminus P_{\text{ge}}$.

Finally we inductively translate any formula $\varphi \in \text{dFO}[2, \Sigma, \{1 \sim_1, 2 \sim_2\}]$ into $T(\varphi) \in \text{dFO}[2, \Sigma \cup \{\text{ge}\}, \{1 \sim_1, 2 \sim_2\}]$ by making it quantify over elements not labeled with **ge**: $T(\sigma(x)) = \sigma(x)$, $T(x \sim_i y) = x \sim_i y$ for $i \in \{1, 2\}$, $T(x = y) = (x = y)$, $T(\exists x. \varphi) = \exists x. \neg \text{ge}(x) \wedge T(\varphi)$, $T(\varphi \vee \varphi') = T(\varphi) \vee T(\varphi')$ and $T(\neg \varphi) = \neg T(\varphi)$.

Lemma 23. *Let φ be a sentence in $\text{dFO}[2, \Sigma]$, $\mathfrak{A} \in \text{Data}[2, \Sigma]$ and $\mathfrak{B} \in \text{Data}[2, \Sigma \cup \{\text{ge}\}]$. The two following properties hold:*

- $\mathfrak{A} \models \varphi$ iff $\mathfrak{A}_{\text{ge}} \models T(\varphi)$
- $\mathfrak{B}_{\setminus \text{ge}} \models \varphi$ iff $\mathfrak{B} \models T(\varphi)$.

Proof. As for any $\mathfrak{A} \in \text{Data}[2, \Sigma]$ we have $(\mathfrak{A}_{\text{ge}})_{\setminus \text{ge}} = \mathfrak{A}$, it is sufficient to prove the second point. We reason by induction on φ . Let $\mathfrak{A} = (A, (P_\sigma)_\sigma, f_1, f_2) \in \text{Data}[2, \Sigma \cup \{\text{ge}\}]$ and let $\mathfrak{A}_{\setminus \text{ge}} = (A', (P'_\sigma)_\sigma, f'_1, f'_2) \in \text{Data}[2, \Sigma]$. The inductive hypothesis is that for any formula $\varphi \in \text{dFO}[2, \Sigma]$ (closed or not) and any context interpretation function $I : \mathcal{V} \rightarrow A'$ we have $\mathfrak{A}_{\setminus \text{ge}} \models_I \varphi$ iff $\mathfrak{A} \models_I T(\varphi)$. Note that the inductive hypothesis is well founded in the sense that the interpretation I always maps variables to elements of the structures.

We prove two cases: when φ is a unary predicate and when φ starts by an existential quantification, the other cases being similar. First, assume that $\varphi = \sigma(x)$ where $\sigma \in \Sigma$. $\mathfrak{A}_{\setminus \text{ge}} \models_I \sigma(x)$ holds iff $I(x) \in P'_\sigma$. As $I(x) \in A \setminus P_{\text{ge}}$, we have $I(x) \in P'_\sigma$ iff $I(x) \in P_\sigma$, which is equivalent to $\mathfrak{A} \models_I T(\sigma(x))$. Second assume $\varphi = \exists x. \varphi'$. Suppose that $\mathfrak{A}_{\setminus \text{ge}} \models_I \exists x. \varphi'$. Thus, there is a $a \in A'$ such that $\mathfrak{A}_{\setminus \text{ge}} \models_{I[x/a]} \varphi'$. By inductive hypothesis, we have $\mathfrak{A} \models_{I[x/a]} T(\varphi')$. As $a \in A' = A \setminus P_{\text{ge}}$, we have $\mathfrak{A} \models_{I[x/a]} \neg \text{ge}(x)$, so $\mathfrak{A} \models_I \exists x. \neg \text{ge}(x) \wedge T(\varphi')$ as desired. Conversely, suppose that $\mathfrak{A} \models_I T(\exists x. \varphi')$. It means that there is a $a \in A$ such that $\mathfrak{A} \models_{I[x/a]} \neg \text{ge}(x) \wedge T(\varphi')$. So we have that $a \in A' = A \setminus P_{\text{ge}}$, which means that $I[x/a]$ takes values in A and we can apply the inductive hypothesis to get that $\mathfrak{A}_{\setminus \text{ge}} \models_{I[x/a]} \varphi'$. So we have $\mathfrak{A}_{\setminus \text{ge}} \models_I \exists x. \varphi'$. $\square$

From Theorem 1, we know that $\text{DATASAT}(\text{dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ is undecidable. From a closed formula $\varphi \in \text{dFO}[2, \Sigma, \{1 \sim_1, 2 \sim_2\}]$, we build the formula $\exists x. \langle\langle T(\varphi) \rangle\rangle_x^3$ which belongs to $\exists\text{-3-Loc-dFO}[2, \Sigma \cup \{\text{ge}\}, \{1 \sim_1, 2 \sim_2\}]$. Now if φ is satisfiable, it means that there exists $\mathfrak{A} \in \text{Data}[2, \Sigma]$ such that $\mathfrak{A} \models \varphi$. By Lemma 23, $\mathfrak{A}_{\text{ge}} \models T(\varphi)$. Let a be an element of $\mathfrak{A}$, then thanks to Lemma 22, we have $\mathfrak{A}_{\text{ge}}|_a^3 \models T(\varphi)$. Finally by definition of our logic, $\mathfrak{A}_{\text{ge}} \models \exists x. \langle\langle T(\varphi) \rangle\rangle_x^3$. So $\exists x. \langle\langle T(\varphi) \rangle\rangle_x^3$ is satisfiable. Now assume that $\exists x. \langle\langle T(\varphi) \rangle\rangle_x^3$ is satisfiable. So there exist $\mathfrak{A} \in \text{Data}[2, \Sigma \cup \{\text{ge}\}]$ and an element a of $\mathfrak{A}$ such that $\mathfrak{A}|_a^3 \models T(\varphi)$. Using Lemma 23, we obtain $(\mathfrak{A}|_a^3)_{\setminus \text{ge}} \models \varphi$. Hence φ is satisfiable. This shows that we can reduce $\text{DATASAT}(\text{dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ to $\text{DATASAT}(\exists\text{-3-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2\})$.

Theorem 10. *The problem $\text{DATASAT}(\exists\text{-3-Loc-dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ is undecidable.*

4.3.2. Radius 2 and three data values. We provide here a reduction from $\text{DATASAT}(\text{dFO}, 2, \{1 \sim_1, 2 \sim_2\})$ to $\text{DATASAT}(\exists\text{-2-Loc-dFO}, 3, \{1 \sim_1, 2 \sim_2, 3 \sim_3\})$. The idea is similar to the one used in the proof of Lemma 19 to show that $\text{DATASAT}(\exists\text{-2-Loc-dFO}, 2, \Gamma_2)$ is N2EXP-hard by reducing the problem $\text{DATASAT}(\text{dFO}, 1, \Gamma_1)$. The intuition is that when we add the same extra value to each node, then all the elements of the structure are in the ball of radius 2 of any element. Indeed we have the following lemma.

Lemma 24. *Let φ be a formula in $\text{dFO}[2, \Sigma, \{1 \sim_1, 2 \sim_2\}]$. There exists $\mathfrak{A} \in \text{Data}[2, \Sigma]$ such that $\mathfrak{A} \models \varphi$ if and only if there exists $\mathfrak{B} \in \text{Data}[3, \Sigma]$ such that $\mathfrak{B} \models \exists x. \langle\langle \varphi \rangle\rangle_x^2$.*

Proof. Assume that there exists $\mathfrak{A} = (A, (P_\sigma)_{\sigma \in \Sigma}, f_1, f_2)$ in $\text{Data}[2, \Sigma]$ such that $\mathfrak{A} \models \varphi$. Consider the 3-data structure $\mathfrak{B} = (A, (P_\sigma)_{\sigma \in \Sigma}, f_1, f_2, f_3)$ such that $f_3(a) = 0$ for all $a \in A$. Let $a \in A$. It is clear that we have $\mathfrak{B}|_a^2 = \mathfrak{A}$ and that $\mathfrak{B}|_a^2 \models \varphi$ (because $\mathfrak{A} \models \varphi$ and φ never mentions the third values of the elements since it is a formula in $\text{dFO}[2, \Sigma, \{1 \sim_1, 2 \sim_2\}]$). Consequently $\mathfrak{B} \models \exists x. \langle\langle \varphi \rangle\rangle_x^2$.

Assume now that there exists $\mathfrak{B} = (A, (P_\sigma)_{\sigma \in \Sigma}, f_1, f_2, f_3)$ in $\text{Data}[3, \Sigma]$ such that $\mathfrak{B} \models \exists x. \langle\langle \varphi \rangle\rangle_x^2$. Hence there exists $a \in A$ such that $\mathfrak{B}|_a^2 \models \varphi$, but then by forgetting the third value in $\mathfrak{B}|_a^2$ we obtain a model in $\text{Data}[2, \Sigma]$ which satisfies φ . $\square$

Using Theorem 1, we obtain the following result.

Theorem 11. *The problem $\text{DATASAT}(\exists\text{-2-Loc-dFO}, 3, \{1 \sim_1, 2 \sim_2, 3 \sim_3\})$ is undecidable.*

5. CONCLUSION

Logic	r	Da	Γ	Decidability status
$\text{dFO}[Da, \Sigma, \Gamma]$	–	0	$\emptyset$	NEXP-complete (Thm 2 [11, 16])
	–	1	$\{1 \sim_1\}$	N2EXP-complete (Thm 3 [17, 30])
	–	2	$\{1 \sim_1, 2 \sim_2\}$	Undecidable (Thm 1 [21])
$r\text{-Loc-dFO}[Da, \Sigma, \Gamma]$	1	2	$\{1 \sim_1, 2 \sim_2, 1 \sim_2\}$	Decidable (Thm 5)
	1	2	$\{1 \sim_1, 2 \sim_2, 2 \sim_1\}$	Decidable (Thm 5)
	1	2	Γ_2	Open problem
	2	2	$\{1 \sim_1, 2 \sim_2\}$	Open problem
	2	2	$\{1 \sim_1, 2 \sim_2, 1 \sim_2\}$	Undecidable (Thm 7)
	2	2	$\{1 \sim_1, 2 \sim_2, 2 \sim_1\}$	Undecidable (Thm 7)
	3	2	$\{1 \sim_1, 2 \sim_2\}$	Undecidable (Thm 6)
$\exists r\text{-Loc-dFO}[Da, \Sigma, \Gamma]$	1	≥ 1	Γ_{Da}	NEXP-complete (Thm 9)
	2	2	Γ_2	N2EXP-complete (Thm 8)
	3	2	$\{1 \sim_1, 2 \sim_2\}$	Undecidable (Thm 10)
	2	3	$\{1 \sim_1, 2 \sim_2, 3 \sim_3\}$	Undecidable (Thm 11)

TABLE 1. Summary of the results for the satisfiability problem

In this work, we have tried to pursue an exhaustive study in order to determine when the satisfiability of the local first order logic with data is decidable. The results we have obtained are provided in Table 1. We observe that even if we consider the existential fragment, as soon as the view of the elements has a radius bigger than 3, the satisfiability problem is undecidable. This table allows us as well to see what are the missing elements in order to fully characterize the decidability status of this satisfiability problem. In particular, we do not know whether the decidability result of Theorem 5 still holds when considering two diagonal relations, but our proof technique does not seem to directly apply. It would be as well very interesting to establish the relative expressive power of these logics and to compare them with some other well-known fragments as for instance the guarded fragment. Finally,

another research direction would be to see how our logic could be used to verify distributed algorithms with data (each element in our model representing a process).

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