Mathematical Programming (MP)

- A general MP formulation

\[
\begin{align*}
\text{(MP)} & \quad \max f(x) \\
\text{s.t.} & \quad x \in X \subseteq \mathbb{R}^n
\end{align*}
\]

- s.t. = *subject to*
- Different types of problems:
  - $X=\mathbb{R}^n$ unconstrained problem
  - $X$ defined by non linear expression or $f(x)$ non linear function $\Rightarrow$ non linear MP problem
Linear Mathematical Programming (LP)

- A MP is LP if:
  - The objective function is linear \( f(x) = c^T x \)

  where \( c^T = [c_1, ..., c_n] \) \( x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \)

  - The set \( X \) is defined by linear equality or inequality constraints

    \[ A \underline{x} \leq \underline{b} \quad \text{where} \quad \underline{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \]
Linear Mathematical Programming (LP)

• An LP problem can be expressed as

\[
(LP) \quad \max f(x) = c^T x
\]

\[
s.t.
\]

\[
A \underline{x} \leq \underline{b}
\]

\[
\underline{x} \geq 0
\]

• The quantities \( c, b, A \)
  are two vectors and a matrix of constant coefficients representing the parameters of the problem
Linear Mathematical Programming (LP)

- The set of feasible solutions of LP can be denoted as
  \[ X = \{ x \in \mathbb{R}^n : A x \leq b, x \geq 0 \} \]

- Types of LP problems:
  \[ X = \{ x \in \mathbb{R}^n : A x \leq b, x \geq 0 \} \quad \text{Continuous LP (LP)} \]
  \[ X = \{ x \in \mathbb{Z}^n : A x \leq b, x \geq 0 \} \quad \text{Integer LP (IP)} \]
  \[ X = \{ x \in \mathbb{R}^{n_1}, y \in \mathbb{Z}^{n_2} : A x + D y \leq b, x \geq 0, y \geq 0 \} \quad \text{Mixed Integer LP (MIP)} \]
Linear Mathematical Programming (LP)

• How can a decision problem to be modeled as a LP problem?
• What corresponds to the set X?
• How can we find a solution to our decision problem?

• We see these aspects considering an example ...
LP – an example

• A company producing paints wants to plan its daily production
• Two types of paints are considered, a paint for interior (I) and a paint for exterior (E)
• Paint production uses two raw materials indicated with A and B
• The daily availability of raw material is: A = 6 tons, B = 8 tons
• The quantity of A and B consumed to produce one ton of paint E and I is known:

<table>
<thead>
<tr>
<th>Raw material</th>
<th>E</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

• We assume that all the paint produced is sold
• The selling price per ton is 3K€ for E and 2K€ for I
• Further information obtained by the company through a market survey:
  – the daily demand of I paint never exceeds more than 1 ton that of paint E
  – the maximum daily demand of paint I is 2-ton
• **Decision problem**: determine the quantities to produce daily for the two paints in order to make the maximize the gain from sales
LP – problem formulation

- **MP formulation of a decision problem**

- **Variables**: define the solution of the problem
- **Objective**: quantifies the quality of a solution (as a function of the variables)
- **Constraints**: identify the set of allowed values for the variables defining the feasible solution set (e.g., technological constraints, budget, operational constraints, etc.)
LP – an example (cont.)

• Problem variables
  – Two variables representing the quantity (tons) of paint of the two types produced (and sold) daily:
    • Production of paint for external: $x_E$
    • Production of paint for internal: $x_I$
  – Continuous and non negative variables

• Problem objective function
  – Maximize the daily gain (K€) from selling the produced paint types

\[ Z = 3x_E + 2x_I \]
  – It is a linear expression
LP – an example (cont.)

- **Problem constraints**
  - Technological constraints: the use of raw materials cannot exceed the material availability
    - Consumption for unit of product
      - \((A)\) \(1x_I + 2x_E \leq 6\)
      - \((B)\) \(2x_E + 1x_I \leq 8\)
    - Availability
    - Raw material
  - Constraints due to the market survey
    - \(x_I - x_E \leq 1\)
    - \(x_I \leq 2\)
  - Variable positivity constraints (variable lower bounds)
    - \(x_E \geq 0\)  \(x_I \geq 0\)
LP – an example (cont.)

• The complete problem formulation

\[
\text{max } Z = 3x_E + 2x_I \\
-x_E + 2x_I \leq 6 \quad (1) \\
2x_E + x_I \leq 8 \quad (2) \\
x_E + x_I \leq 1 \quad (3) \\
x_I \leq 2 \quad (4) \\
x_E \geq 0 \quad (5) \\
x_I \geq 0 \quad (6)
\]

• It is a LP problem
LP – Graphic solution

The \((x_E, x_I)\) plane

Non negativity constraints

\[ x_E \geq 0 \]  \hspace{1cm} (5)

\[ x_I \geq 0 \]  \hspace{1cm} (6)

Introducing the constraints

\[ x_E + 2x_I \leq 6 \]  \hspace{1cm} (1)

\[ 2x_E + x_I \leq 8 \]  \hspace{1cm} (2)

\[ -x_E + x_I \leq 1 \]  \hspace{1cm} (3)

\[ x_I \leq 2 \]  \hspace{1cm} (4)
LP – Polyedra

- The feasibility set $X$ includes the inner points and the sides (in case of strictly inequality the problem is non linear)
- $X$ is a Polyedron obtained from the intersection of the semi-spaces defined by hyperplanes (lines in the example)
- What’s the difference between LP and IP polyedra?

- A polyedron for an LP problem include $\infty$ feasible solutions

$$X = \{ x \in \mathbb{R}^n : A x \leq b \}$$
The polyhedron for an IP problem has a finite number of feasible solutions

\[ X = \{ x \in \mathbb{Z}^n : A x \leq b \} \]

Red dots on the sides and in the polyhedron correspond to the integer feasible solutions.
LP - Graphic solution

The obj function is drawn in the plane \((x_E,x_I)\) as a parametric line that depends on the value fixed for \(Z\)

\[ Z = 3x_E + 2x_I \]

- If \(Z=0\)
  \[ 3x_E + 2x_I = 0 \]
- If \(Z=6\)
  \[ 3x_E + 2x_I = 6 \]

Operations Research – Massimo Paolucci – DIBRIS University of Genova
LP - Graphic solution

- The problem solution (the value pair of value for \((x_E, x_I)\) that maximizing the objective function) can be found graphically by shifting as much as possible to the right (increasing \(Z\) since we want to maximize) this function so that it has an intersection with the region \(X\).

- The points on the perimeter of \(X\) then play a fundamental role: these points are the **vertices** of the polyhedron.
LP - Graphic solution

A=(0,0)  
B=(4,0)  
C=(10/3, 4/3)  
D=(2,2)  
E=(1,2)  
F=(0,1)  

\[ A=(0,0) \]
\[ B=(4,0) \]
\[ C=(\frac{10}{3}, \frac{4}{3}) \]
\[ D=(2,2) \]
\[ E=(1,2) \]
\[ F=(0,1) \]

Optimal solution
Z=0
Z=6
Z=\frac{38}{3}
Property:
The optimal solution lies on the frontier of polyedron $X$

$$Z^* = 3x_E + 2x_I = \frac{38}{3}$$

Optimal solution $C = (10/3, 4/3)$
LP - Graphic solution

- Linearity implies that optimal (finite) solutions are in the polyedron vertices.
- More than a single (finite) equivalent solutions (infinite number) if the objective function is parallel to a constraint.
- Even in these cases the search for optimal solution is limited to polyedron vertices.

All the points on DC are optimal.
In the paint production problem it can be interesting to know how the raw materials (available resources) are used:

- Is there any advantage from an increase of the resource availability?
- Is it possible to decrease the resource availability without changing the optimal objective value?

Another interesting point is to analyse how the optimal solution would change if the selling prices change.
LP - Graphic solution and post-optimality

• Post-optimality $\Leftrightarrow$ Sensitivity analysis w.r.t changes in resources availability

All the constraints are $\leq$ so they can be viewed as

Quantity of resource used $\leq$ Resource availability

– Constraints (1) and (2) are satisfied as equalities by the optimal solution
– Then both raw materials A and B are used up to their availability
– Constraints (1) and (2) are **saturated** and raw materials A and B are said **scarce resources**
First case: increase the availability of a scarce resource
What is the maximum increment that is worth considering for resource A?

Increasing resource A availability moves constraints (1) and so the optimal point

K=(3,2)
LP - Graphic solution and post-optimality

Increase beyond $K=(3,2)$ is no profitable (why?)

Availability of A increases from 6 to 7

The new (1')

$$x_E + 2x_I \leq 7 \quad (1')$$
Same behaviour for resource B

Increasing availability of B move (2) and optimal point

Beyond \( L=(6,0) \) no reason for further increase of B.

New availability for B is 12

\[ 2x_E + x_I \leq 12 \]  \hspace{1cm} (2')
Opposite case: decrease of abundant resources
Assume (3) and (4) associated with (abundant) resources
How much can we reduce their availability without affecting the current optimal solution?

Constraint (3)

\[-x_E + x_I \leq 1\]
Opposite case: decrease of abundant resources
Assume (3) and (4) associated with (abundant) resources
How much can we reduce their availability without affecting the current optimal solution?

Constraint (4)
LP - Graphic solution and post-optimality

Which among resource A and B is worth increasing first? (the company may have a limited budget to invest)

• The *Unit Value of a resource* $y_i$ (also called *Shadow Price*):

$$y_i = \frac{\max Z \text{ variation}}{\max \text{ resource } i \text{ variation}}$$

• $y_i = \text{ the increase of objective for a unitary increase of resource } i \text{ availability}$
LP - Graphic solution and post-optimality

Examples

• Resource A:

\[ y_A = \frac{13 - \frac{38}{3}}{7 - 6} = \frac{39 - 38}{3} = \frac{1}{3} \quad (K\€/ton) \]

• Resource B:

\[ y_B = \frac{18 - \frac{38}{3}}{12 - 8} = \frac{54 - 38}{4} = \frac{4}{3} \quad (K\€/ton) \]

• Resource B is the most convenient to increase
LP - Graphic solution and post-optimality

Consider a possible variation in the product selling prices
How the optimal solution change?

• Changing coefficient $c_E$ and $c_I$ the slope of objective function changes

The slope changes as:

$$\frac{c_E}{c_I} = -\frac{3}{2}$$
Changing $c_E$ e $c_I$ point C remains the optimal solution until the slope of the objective function equals the one of either (1) or (2).

$c_I$ decrease
$c_E$ increase

(2)
LP - Graphic solution and post-optimality

• Compute the ranges for $c_E$ and $c_I$

  **First case:** change $c_E$ with fixed $c_I = 2$

  $-\frac{c_E}{2} = -\frac{1}{2}$ = slope of (1)  $\Rightarrow c_E = 1$

  $-\frac{c_E}{2} = -2$ = slope of (2)  $\Rightarrow c_E = 4$

  $\Rightarrow 1 \leq c_E \leq 4$

  – If $c_E = 1$ C and D are optimal. If $c_E$ becomes < 1 only D is optimal
  – If $c_E = 4$ C and B are optimal. If $c_E$ becomes > 4 only B is optimal
LP - Graphic solution and post-optimality

• Compute the ranges for $c_E$ and $c_I$

  **Second case:** change $c_I$ with fixed $c_E = 3$

  $-\frac{3}{c_I} = -\frac{1}{2} = \text{slope of (1)} \Rightarrow c_I = 1$

  $-\frac{3}{c_I} = -2 = \text{slope of (2)} \Rightarrow c_I = \frac{3}{2}$

  \[
  \begin{cases}
  -\frac{3}{c_I} = -\frac{1}{2} \\
  -\frac{3}{c_I} = -2
  \end{cases}
  \Rightarrow \frac{3}{2} \leq c_I \leq 6
  \]

  - If $c_I = 3/2$ C and B are optimal. If $c_I$ becomes $< 3/2$ only B is optimal
  - If $c_I = 6$ C and D are optimal. If $c_I$ becomes $> 6$ only D is optimal