The cutting planes method

The idea:

– If the solution of the linear relaxation (RL) of a problem (IP) is not integer then the optimal integer solution lies in the polyhedron $P$ of (RL)
– We can add constraints to $P$ in order to tighten it
– These new constraints cut parts of the polyhedron
– The cut parts do not contain integer solutions
– The optimal integer solution is sought by solving a sequence of relaxed problems progressively adding cuts
The cutting planes method

Let (IP) \( \max x_0 = \mathbf{c}^T x \)
\[ Ax = b \quad \Rightarrow \quad P = \{ x \in \mathbb{Z}^n : Ax = b, x \geq 0 \} \]
\[ x \in \mathbb{Z}_+^n \]

Let \( x^* \) the optimal integer solution of (IP)

The first (RL) problem with \( x^* \) optimal solution

(RL) \( \max x_0 = \mathbf{c}^T x \)
\[ Ax = b \quad \Rightarrow \quad P_0 = \{ x \in \mathbb{R}^n : Ax = b, x \geq 0 \} \]
\[ x \in \mathbb{R}_+^n \]

A sequence of polyhedra is build (Gomory sequence) such that

\[ P_0 \supset P_1 \supset \ldots \supset P_t \]
\[ P_i \cap \mathbb{Z}^n = P \]
\[ x^* \notin P_i \]
\[ x_i^* \equiv x^* \]
The cutting planes method

The sequence is obtained adding cuts to $P_0$

**Definition**

An inequality $a^T x \geq a_0$ is a **cut** for polyhedron $P'$ of the relaxation (RL) of problem (IP) if, being $x^\circ$ the optimal non integer solution of (RL), it holds:

1) $a^T y \geq a_0 \ \forall y \in P$  *(valid inequality)*
2) $a^T x^\circ < a_0$  *(not satisfied by $x^\circ$)*

- The cutting planes method finds the optimal integer solution by adding a finite number of cuts
- Each cut separates the non integer solution of the current (RL) from the feasible solutions for (IP)
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Example

Non integer optimum for (RL)

Non valid inequality: non integer optimum of (RL) not cut off

Valid inequality

Non valid inequality: two (IP) feasible solutions cut off
The cutting planes method: fractional cut

• Defined by R. Gomory (1950)
• Let assume to have found the (non integer) optimum to (RL) of (IP)
• The $m$ basic variables and the solution

\[ x_{B_i} = y_{i0} - \sum_{j \in R} y_{ij} x_j \quad i = 1, \ldots, m \]  

(1)

\[ x_{B_i} = y_{i0} \quad i = 1, \ldots, m \quad x_j = 0 \quad \forall j \in R \]

• Let $i$ a non integer component

\[ y_{i0} = \lfloor y_{i0} \rfloor + f_{i0} \quad 0 < f_{i0} < 1 \]

\[ y_{ij} = \lfloor y_{ij} \rfloor + f_{ij} \quad 0 \leq f_{ij} < 1 \]

\[ \lfloor a \rfloor = \text{floor} = \text{the largest integer not greater than} \ a \]
The cutting planes method: fractional cut

Rewriting l.h.s. of \( i \)-th equation (1)

\[
y_{i0} = x_{B_i} + \sum_{j \in R} \lfloor y_{ij} \rfloor x_j + \sum_{j \in R} f_{ij} x_j \geq x_{B_i} + \sum_{j \in R} \lfloor y_{ij} \rfloor x_j \geq 0
\]

then imposing that \( x_{B_i} \) is integer

\[
x_{B_i} + \sum_{j \in R} \lfloor y_{ij} \rfloor x_j \leq y_{i0} = \lfloor y_{i0} \rfloor + f_{i0} \implies x_{B_i} + \sum_{j \in R} \lfloor y_{ij} \rfloor x_j \leq \lfloor y_{i0} \rfloor
\]

integer

Substituting \( x_{B_i} \) from (1)

\[
\sum_{j \in R} \lfloor y_{ij} \rfloor x_j + \sum_{j \in R} f_{ij} x_j - \lfloor y_{i0} \rfloor - f_{i0} - \sum_{j \in R} \lfloor y_{ij} \rfloor x_j \geq -\lfloor y_{i0} \rfloor
\]
The cutting planes method: fractional cut

- Then the Gomory’s Fractional Cut is

\[
\sum_{j \in R} f_{ij} x_j \geq f_{i0} \quad (2)
\]

**Theorem**

For every non integer component \( i \) of the non integer solution of (RL) the inequality (2) is a cut for polyhedron \( P \)
The cutting planes method: fractional cut

Example

\[ \text{max } \begin{align*} x_0 & = 2x_1 + x_2 \\ x_1 + x_2 & \leq 5 \quad (1) \\ -x_1 + x_2 & \leq 0 \quad (2) \\ 6x_1 + 2x_2 & \leq 21 \quad (3) \\ x_1, x_2 & \geq 0 \\ x_1, x_2 & \in \mathbb{Z} \end{align*} \]

\[ \text{max } x_0 = 2x_1 + x_2 \]
\[ x_1 + x_2 + x_3 = 5 \]
\[ -x_1 + x_2 + x_4 = 0 \]
\[ 6x_1 + 2x_2 + x_5 = 21 \]
\[ x_1, x_2, x_3, x_4, x_5 \geq 0 \]
\[ x_1, x_2 \in \mathbb{Z} \]

Solving the (RL)

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<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( x_1 )</th>
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The Gomory’s cut from \( x_2 \) row
The cutting planes method: fractional cut

The associated equation

\[ x_2 + \frac{3}{2} x_3 - \frac{1}{4} x_5 = \frac{9}{4} \quad \Rightarrow \quad x_2 = \frac{9}{4} - \left( \frac{3}{2} x_3 - \frac{1}{4} x_5 \right) \]

\[ y_{ij} = \lfloor y_{ij} \rfloor + f_{ij} \quad 0 \leq f_{ij} < 1 \]

\[ 0 < f_{i0} < 1 \]

The fractional cut

\[ \frac{1}{2} x_3 + \frac{3}{4} x_5 \geq \frac{1}{4} \]
The cutting planes method: fractional cut

\[ \frac{1}{2} x_3 + \frac{3}{4} x_5 \geq \frac{1}{4} \]

substituting \( x_1 \) and \( x_2 \)

\[ 5x_1 + 2x_2 \leq 18 \]

Not satisfied by \((11/4, 9/4)\)