An application of Integer Linear Programming: the Sudoku game
We shall see how to formulate the well-known Sudoku game as an Integer Linear Programming problem, identifying the decision variables, the constraints, and the objective function.
The Sudoku rules are as follows:

- A table with 9 rows and 9 columns is given.

- The table is divided into 9 square subtables of dimensions $3 \times 3$.

- Each subtable has to be filled with the numbers \{1, 2, 3, 4, 5, 6, 7, 8, 9\} according to the following rules:
  - each box of the table must contain one and only one number;
  - each number must appear once and only once
    - in each row of the table,
    - in each column of the table,
    - in each subtable;
  - the numbers contained in some boxes of the table are specified \textit{a priori}. 


Choice of the decision variables

- \( x_{ijk} \in \{0, 1\} \).

- The indices \( i, j, \) and \( k \) range from 1 to 9.

- \( x_{ijk} \) is equal to 1 if and only if the number \( i \) appears in the table at the intersection of the \( j \)-th row and the \( k \)-th column; otherwise it is equal to 0.
Description of the constraints

- Each box of the table must contain one and only one element:
  \[ \forall j, k \in \{1, \ldots, 9\}, \]
  \[ \sum_{i=1}^{9} x_{ijk} = 1. \]

- Each number must appear once and only once in each row of the table:
  \[ \forall i, j \in \{1, \ldots, 9\}, \]
  \[ \sum_{k=1}^{9} x_{ijk} = 1. \]
• Each number must appear once and only once in each column of the table:

\[ \forall i, k \in \{1, \ldots, 9\}, \]

\[ \sum_{j=1}^{9} x_{ijk} = 1. \]

• Each number must appear once and once in each subtable:

\[ \forall i \in \{1, \ldots, 9\}, \forall m, n \in \{1, 2, 3\}, \]

\[ \sum_{j=3m-2}^{3m} \sum_{k=3n-2}^{3n} x_{ijk} = 1. \]
Some elements of the table are specified *a priori*. In the example considered:

\[
\begin{align*}
    x_{153} &= x_{166} = 1, \\
    x_{216} &= x_{231} = x_{299} = 1, \\
    x_{311} &= x_{339} = x_{342} = x_{385} = x_{398} = 1, \\
    x_{471} &= x_{484} = 1, \\
    x_{522} &= x_{588} = 1, \\
    x_{619} &= x_{626} = x_{655} = x_{694} = 1, \\
    x_{744} &= x_{779} = 1, \\
    x_{825} &= x_{857} = x_{891} = 1, \\
    x_{912} &= x_{968} = 1.
\end{align*}
\]
• In this example, the sole interest is to identify an admissible solution to the problem, i.e., a choice of the decision variables that meets all the constraints outlined above.

• Therefore, an integer linear programming problem can be formulated by choosing any linear objective function.
  • In the following, we will choose as objective function $z = 0$, which has the advantage of not preferring any admissible solution to any other admissible solution.
Summing up: Formulation of the Sudoku as an IP problem

\[ \begin{align*}
\text{min} & \quad z = 0 \\
\text{s.t.} & \quad \forall j, k \in \{1, \ldots, 9\}, \sum_{i=1}^{9} x_{ijk} = 1, \\
& \quad \forall i, j \in \{1, \ldots, 9\}, \sum_{k=1}^{9} x_{ijk} = 1, \\
& \quad \forall i, k \in \{1, \ldots, 9\}, \sum_{j=1}^{9} x_{ijk} = 1, \\
& \quad \forall i \in \{1, \ldots, 9\}, \forall m, n \in \{1, 2, 3\}, \sum_{j=3m-2}^{3m} \sum_{k=3n-2}^{3n} x_{ijk} = 1,
\end{align*} \]
\[ x_{153} = x_{166} = 1, \]
\[ x_{216} = x_{231} = x_{299} = 1, \]
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\[ x_{744} = x_{779} = 1, \]
\[ x_{825} = x_{857} = x_{891} = 1, \]
\[ x_{912} = x_{968} = 1, \]
\[ \forall i, j, k \in \{1, \ldots, 9\} \text{, } x_{ijk} \in \{0, 1\}. \]
Solution of the problem

- Solution using a solver that employs IP (http://sudoku.noisette.ch/index.php).

Sudoku to resolve

```
  3  9   
  5   6
  2   

  1  6   
  3  7
  8   

  4  3   
  4  1
  5   

  8  6   
```

6  3  2
Solution of the problem (continued)

- **Solution** obtained via the solver:

```
Solved sudoku

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A method to verify the uniqueness or not of the solution

- In this specific problem and with the choice made for the objective function, the set of admissible solutions coincides with the set of optimal solutions.

- Let $y_{ijk} := x_{ijk}^*$ be the components of an admissible solution, found by solving the previous problem. In order to verify whether or not it is the only admissible solution of Sudoku, a new constraint is introduced into the problem, which excludes from the set of admissible solutions the solution that has just been found:

\[
\sum_{i=1}^{9} \sum_{j=1}^{9} \sum_{k=1}^{9} y_{ijk} x_{ijk} \leq 9 \cdot 9 - 1 = 80.
\]

- Indeed, remembering that $x_{ijk} \in \{0,1\}$, the previous sum is equal to 81 if and only if $x_{ijk} = y_{ijk}$ for every $i$, $j$ and $k$. 